Do five of the following seven problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions, including one from the ODE part (Questions 1-3) and one from the PDE part (Questions 4-7). Please write your work clearly justifying when necessary what you use.

(1) Consider the matrix

\[ A = \frac{1}{9} \begin{pmatrix} 5 & 4 & 2 \\ -2 & 11 & 1 \\ -4 & 4 & 11 \end{pmatrix}, \]

all of whose eigenvalues are 1.

(a) Find the semisimple-nilpotent decomposition \( A = S + N \) of \( A \).

(b) Use this to calculate the matrix exponential and give the general solution of the system

\[ \frac{dy}{dt} = Ay. \]

(2) Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) is locally Lipschitz, and you are given a non-negative \( C^1 \) function \( L : \mathbb{R}^n \to \mathbb{R} \) satisfying \( L(x) \to \infty \) as \( \|x\| \to \infty \), and with the property

\[ \langle \nabla L(x), f(x) \rangle \leq c_1 + c_2 L(x) \quad \text{for all } x, \]

where \( c_i \) are non-negative constants. If \( x(t) \) solves the ODE

\[ \dot{x} = f(x), \quad \text{with } x(t_0) = x_0, \]

find an integral inequality for \( L(x(t)) \) and use it to show that \( L(x(t)) \) remains bounded for all \( t \). Conclude that \( \|x\| \) remains finite and thus the ODE has a globally defined solution.
(3) Consider the nonlinear system

\[\begin{align*}
    x' &= -\lambda(r) \, x + \omega(r) \, y \\
    y' &= -\omega(r) \, x - \lambda(r) \, y
\end{align*}\]

where \(r = \sqrt{x^2 + y^2}\) and \(\lambda\) and \(\omega\) are given smooth functions of \(r \geq 0\).

(a) Determine the stability of the rest point \((x, y) = (0, 0)\) in terms of \(\lambda(0)\) and \(\omega(0)\) and describe the qualitative behavior near the origin. Assume that \(\omega(0) \neq 0\) and that \(\frac{d\lambda}{dr}(0) \neq 0\) if \(\lambda(0) = 0\).

(b) Suppose now that

\[\begin{align*}
    \lambda(r) &= r \, (1 - r) \, (2 - r) \\
    \omega(r) &= \left(\frac{1}{2} - r\right) \, (\frac{3}{2} - r).
\end{align*}\]

Describe all periodic orbits and limit sets of the system, and sketch the phase plane.

(4) (a) Determine the type (elliptic, parabolic or hyperbolic) of the equation

\[u_{xx} + 5u_{xy} + 6u_{yy} = 0.\]

Find the characteristic curves, reduce to canonical form and find the general solution.

(b) Use the method of characteristics to find the solution \(u = u(x, y)\) of

\[\begin{align*}
    x^2u_x + xyu_y &= u^2, \\
    u(y^2, y) &= 1.
\end{align*}\]

Determine whether and where the solution becomes singular.

(5) Prove that

\[K(|x|) = -\frac{1}{4\pi} \frac{\cos(k|x|)}{|x|}, \quad |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}\]

is a fundamental solution for \(\Delta + k^2\) on \(\mathbb{R}^3\).

Hints: If \(\phi \in C_0^\infty(\mathbb{R}^3)\) then for large enough \(R\) and small \(\varepsilon > 0\), apply Green’s identity on the set \(\Omega_\varepsilon = B_R(0) \setminus B_\varepsilon(0)\). Argue that on the interior sphere \(|x| = \varepsilon\), we have

\[\frac{\partial K}{\partial n} = -\frac{dK}{dr}\bigg|_{r=\varepsilon} = -\frac{1}{4\pi\varepsilon} \left(k\sin(k\varepsilon) + \cos(k\varepsilon)\right).\]

You may use without proof that \(K(|x|) = -\frac{\cos(k|x|)}{4\pi|x|}\) is integrable at \(x = 0\).
(6) Let $S := \{(x, y) : -1 \leq x, y \leq 1\}$ be the unit square, and $f : S \to \mathbb{R}$ be a smooth function on $S$. Prove that any smooth solution $u(x, y, t)$ on $S \times [0, \infty)$ of the equation
\[
\begin{aligned}
    u_t &= \Delta u + uu_x + uu_y & \text{in} & \quad S \times (0, \infty) \\
    u(x, y, 0) &= f(x, y) & \text{for all} & \quad (x, t) \in S
\end{aligned}
\]
satisfies the **weak maximum principle**:
\[
\max_{S \times [0, T]} u(x, y, t) \leq \max\left\{ \max_{0 \leq t \leq T} u(\pm 1, \pm 1, t), \max_{(x, y) \in S} f(x, y) \right\}
\]
for any fixed $T > 0$.

**Hint:** Consider $u = v + \varepsilon t$ for $\varepsilon > 0$ and analyze the various cases for $v$ to have a maximum.

(7) Let $\Omega \subset \mathbb{R}^n$, $T > 0$ and $u = u(x, t)$ be a smooth solution to the following initial boundary value problem
\[
\begin{aligned}
    u_{tt} - \Delta u + u^3 &= 0 & \text{in} & \quad \Omega \times [0, T] \\
    u(x, t) &= 0 & \text{for all} & \quad (x, t) \in \partial \Omega \times [0, T]
\end{aligned}
\]

(a) Derive an **energy equality** for $u$.

(b) Show that if $u(x, 0) = 0 = u_t(x, 0)$ for $x \in \Omega$, then $u \equiv 0$. 