(1) Let $X$, $Y$ be topological spaces with $Y$ compact. Let $p: X \times Y \to X$ be projection onto the first factor. Show that $p$ maps each closed set in $X \times Y$ to a closed set in $X$.

(2) Give an example of a locally connected space $X$ and a continuous surjective map $f: X \to Y$ such that $Y$ is not locally connected.

(3) Let $X = [0, 3]/(1, 2)$ be the quotient space of the closed interval $[0, 3]$ with the open interval $(1, 2)$ identified to a point. Let $f: X \to \mathbb{R}$ be a continuous map.
   (a) Prove that $f$ achieves a global minimum.
   (b) Prove that $X$ is not Hausdorff.

(4) Let $X \subset \mathbb{R}^n$ be a subspace. Let $F: \mathbb{R}^n \to Y$ be a continuous map and $f = F|_X$ be the restriction. Show that the induced homomorphism

$$f_*: \pi_1(X, x) \to \pi_1(Y, f(x))$$

is trivial (sends everything to the identity element).

(5) Let $\{A_\alpha\}$ be a collection of subsets of $X$ such that $X = \bigcup A_\alpha$. Let $f: X \to Y$ and suppose that the restrictions $f|_{A_\alpha}$ are all continuous.
   (a) Show that if the collection $\{A_\alpha\}$ is finite and each $A_\alpha$ is closed, then $f$ is continuous.
   (b) Find an example where the collection is countably infinite and each $A_\alpha$ is closed, but $f$ is not continuous.

(6) Recall that $g: X \to Y$ is called proper if $g^{-1}(C)$ is compact whenever $C \subset Y$ is compact. Show that if a (not necessarily continuous) map $f: X \to Y$ is closed and $f^{-1}(y)$ is compact for all $y \in Y$, then $f$ is proper.
Let $X$ be the infinite 3-valent tree. Thus $X$ is an infinite graph containing no cycle, and such that every vertex is incident to three edges. Let $x_0$ be a fixed vertex. Give $X$ a metric $d$ by identifying each edge and its two vertices with the closed interval $[0, 1] \subset \mathbb{R}$ endowed with the usual Euclidean metric, and give $X$ the metric topology. For each nonnegative integer $n$ let $X_n$ be the closed subset $\{x \in X \mid d(x, x_0) \leq n\}$. (See the figure for a picture of $X_3$.)

(a) Prove that each $X_n$ is compact.
(b) Prove that each $X_n$ is contractible.
(c) Prove that $X$ is simply-connected.

**Figure 1.** The subset $X_3$ of the infinite 3-valent tree.