Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master’s level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Compute the integral
\[ \int_0^\infty \frac{x \sin 3x}{(x^2 + 1)(x^2 + 4)} \, dx. \]
Justify your answer carefully.

2. Consider the meromorphic function
\[ f(z) = \frac{1}{z^2 - (2 + 5i)z + 10i}. \]
In each of the following cases, compute the Laurent series
\[ f(z) = \sum_{n \in \mathbb{Z}} c_n(z - a)^n \]
of \( f \) centered at \( a \) which is valid in a neighbourhood of \( b \), and determine its domain of convergence
(i) \( a = b = 2 \).
(ii) \( a = 0, \ b = 3 \).

3. For each \( n \geq 3 \), find the number of zeros (counting multiplicities) of \( z^n + 3z + 1 \) in the annulus \( A = \{1 < |z| < 2\} \). Determine whether these zeros are all simple.

4. Let \( U \) be the portion of the open unit disk given in polar coordinates by
\[ U := \{ re^{i\theta} : 0 < r < 1, \text{ and } 0 < \theta < \pi/3 \}. \]
The boundary of \( U \) consists of the line segment \( L_0 \) from 0 to 1, the line segment \( L_{\pi/3} \) from 0 to \( e^{\pi i/3} \), and a curve \( \Gamma \) on the unit circle. Prove that there exists a unique fractional linear transformation \( f \) satisfying \( f(1) = i, \ f(e^{\pi i/3}) = 0, \ f \) maps \( \Gamma \) into the imaginary line \( \mathbb{R}i \), and \( f \) maps \( L_{\pi/3} \) into the real axis. Give an explicit, simple formula for \( f(z) \). Justify your answer. Hint: Find \( f^{-1}(\infty) \) first.
5. Show that if \( f(z) \) is meromorphic in the extended complex plane \( \mathbb{C} \cup \{\infty\} \) then \( f \) is a rational function.

6. State and prove Liouville’s Theorem for entire functions.

7. Evaluate the following integrals, where \( C = \{z : |z| = 4\} \) traversed once in the counterclockwise direction.

   (a) \( \int_C \frac{z^4}{e^z + 1} \, dz \).

   (b) \( \int_C \frac{z^3 \cos(1/z)}{z^4 + 1} \, dz \).

8. Prove that the series

\[
\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}
\]

defines an analytic function \( f(z) \) in the open set \( \mathbb{C} \setminus \mathbb{Z} \). Prove also that \( f \) has an antiderivative on \( U \).

9. Calculate

\[
\int_0^\pi \frac{dx}{2 + \cos^2(x)}.
\]

10. Let \( f \) be a one-to-one holomorphic map from a region \( \Omega_1 \) onto a region \( \Omega_2 \). Assume that the closure of the unit disc \( D := \{z : |z| < 1\} \) is contained in \( \Omega_1 \). Prove that the inverse function \( f^{-1} : f(D) \to D \) is given by the integral formula

\[
f^{-1}(\omega) = \frac{1}{2\pi i} \int_{\{z=1\}} \frac{f'(z) \cdot z}{f(z) - \omega} \, dz
\]