DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
ADVANCED EXAM - Probability and Multivariate Distribution Theory
AUGUST 2011

Work all problems. 70 points are required to pass with at least 25 from each part.

Part I: Advanced Probability

1. (20 Points) State and prove a version of the Central Limit Theorem. You may assume the existence of a moment generating function.

2. (20 Points) Suppose X and Y are continuous random variables with $E(|X|) < \infty$ and $E(|Y|) < \infty$.

   (a) Prove that $E(X) = E\{E(X|Y)\}$.

   (b) Prove that $var(X) = E\{var(X|Y)\} + var\{E(X|Y)\}$.

   (c) Suppose we want to estimate $E(Y_i)$. The data are $D_1Y_1, D_2Y_2, \ldots$ where $D_i \in \{0, 1\}$, and if $D_i = 0$ then Y_i is unobserved. $D_i = 1$ means that Y_i was observed. Additionally, there are "covariates" (X_is) available, and we know $p(X_i) = E(D_i|X_i)$. Further assume that Y_i is independent of $D_i|X_i$. Prove that $E\{D_iY_i/p(X_i)\} = E(Y_i)$.

3. (10 Points) Prove the second Borel-Cantelli lemma: If the events $\{A_n\}$ are independent then $P(A_n) = 1$ implies $P(\omega : \omega \in A_n i.o.) = 1$. where i.o. stands for infinitely often.

Part II: Linear Models

4. (20 Points) Let $N_p(\mu, \Gamma)$ denote the p-dimensional normal with mean vector μ and dispersion (variance/covariance) matrix Γ.

   (a) Let $X \sim N_p(\mu, \Gamma)$ where Γ is positive definite. Justify that if a is any p by 1 vector, then
   $$\frac{a'(X-\mu)}{(a'\Gamma a)^{1/2}} \sim N(0, 1).$$

   (b) Let Y be a random vector in R^n which is independent of X defined above. Show that if $\Pr(Y'\Gamma Y = 0) = 0$, then
   $$\frac{Y'(X-\mu)}{(Y'\Gamma Y)^{1/2}} \sim N(0, 1).$$

   (Hint: define cdf of the statistic as an indicator variable, the apply double expectation formula.)

(c) If X_1, X_2 and X_3 are i.i.d. $N(0,1)$. Justify that

$$\frac{X_1 e^{X_3} + X_2 \log|X_3|}{(e^{2X_3} + (\log|X_3|)^2)^{1/2}} \sim N(0,1).$$

5. (20 Points) Let $Y \sim N_p(\mu, I)$. Let A and B be $k \times p$ matrices. Suppose and $AB' = 0$.

   (a) Prove that AY and BY are independent.

   (b) Suppose $A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$. Show that AY and BY are independent only if $|a| = 1/|b|$. What are possible choices for a and b?

   (c) Prove that $AB' = 0$ implies that $Y'AY$ and $Y'BY$ are independent. (You may use the form for A and B in part (b).)

   (d) Derive the expectation of $Y'AY$.

6. (10 Points) Let $X \sim N_p(\mu, \Gamma)$ with Γ being positive definite. Let A be a $p \times p$ symmetric scalar matrix with rank r. State a theorem about a necessary and sufficient condition for $X'AX$ to be a chi-square distribution, specifying associated parameters.