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UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM - DIFFERENTIAL EQUATIONS
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Do five of the following problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

(1) Suppose that $x(t)$ satisfies the equation $x' = Ax$ where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & -2 & 0 \end{pmatrix},$$

where β is a parameter within the range $-\infty < \beta < 1$. Calculate the general, real-valued solution and the stable, center, and unstable solution spaces for all values of $\beta < 1$.

(2) Suppose that $f : R^n \to R^n$ is a C^∞ vector field. Let $x(t, x_o)$ denote the solution of the initial value problem

$$\frac{dx}{dt} = f(x)$$
$$x(0, x_o) = x_o.$$

(a) Consider the change of independent variable $t \to s$ where

$$s = s(t, x_o) = \int_0^t (1 + |f(x(r, x_o))|^2)\, dr,$$

and $|\cdot|$ is the Euclidean norm, and define a change of dependent variable implicitly by the equation $y(s, x_o) = x(t, x_o)$. Determine the initial value problem satisfied by y.

(b) Prove that every solution $y(s, x_o)$ of the initial value problem in (a) is defined and continuously differentiable on the interval $-\infty < s < \infty$.

(3) Consider the following system

$$x' = -2x + y$$
$$y' = x^2 - y^2 + 3$$

(a) Determine all rest points and their linearized stability.
(b) Prove the existence of a connecting orbit between the two rest points. Sketching the phase plane is useful but is *not* in itself a complete answer to this question. You need to present a mathematical argument that justifies your sketch.

(4) Suppose that $u(t)$ and $\bar{u}(t)$ are two solutions on the interval $0 \leq t \leq T$ of the system

$$u' = f(u),$$

where $n \geq 2$ and $f : R^n \to R^n$ is a smooth vector field and that

$$\frac{\partial f_i}{\partial u_j}(u) > 0$$

holds for each $j \neq i$ and all u.

Prove that if $u(t)$ and $\bar{u}(t)$ satisfy the initial inequalities $u_i(0) < \bar{u}_i(0)$, then the two solutions satisfy

$$u_i(t) < \bar{u}_i(t)$$

on the interval $0 \leq t \leq T$ for all i.

(5) Consider the biharmonic equation

$$\Delta^2 u = f \quad \text{in} \quad \Omega, \quad \text{with data}$$
$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a regular bounded open subset of the plane R^2, and $f \in L^2$.

(a) Define a weak solution for this problem. Be sure to explain your definition.
(b) Show that any weak solution which is also a C^4 function is a classical solution.

(6) Consider the equation

$$u_t + \Delta(\epsilon \Delta u - \beta u) = 0,$$

defined on the torus $\mathbb{T}^3$, where ϵ and β are positive constants.

(a) Show that

$$E(t) = \frac{\epsilon}{2} \int |\nabla u|^2 \, dx + \frac{\beta}{2} \int u^2 \, dx$$

is monotonically decreasing for all C^4 solutions u.

(b) Show that the IVP

$$u_t + \Delta(\epsilon \Delta u - \beta u) = f(x,t)$$
$$u(x,0) = g(x)$$

has at most one C^4 solution on the torus, for any smooth f and g.

(7) Consider the heat equation $u_t = \epsilon u_{xx}$ on the halfplane $t > 0$, with initial data

$$u_0(x) = \begin{cases} 1 & x < 0, \\ 0 & x > 0. \end{cases}$$

Find an integral representation for the solution $u(t,x)$, and use this to show that $u(t,x) \to c$ as $t \to \infty$ for each fixed x, for some constant c. Is this convergence uniform? If so, give a proof; if not, explain why not.