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Advanced Exam – Algebra. August 29, 2011.

Passing Standard: It is sufficient to do five problems correctly, including at least one problem from each of the three parts.

1. Group Theory and Representation Theory

1. Let G be a finite Abelian group (written additively) of odd order $2k+1$. Let $\tau : G \to G$ be an automorphism of order 2 defined by formula

$$\tau(x) = -x.$$

Let $\hat{G}$ be a semidirect product of $\mathbb{Z}_2$ and G defined using τ.

- **a.** Find the number of conjugacy classes in $\hat{G}$.
- **b.** Find the number of irreducible representations of $\hat{G}$ and their dimensions.

2. Let G be a group (not necessarily finite) and let $V = \mathbb{C}[G]$ be a complex vector space with a basis $\{e_g : g \in G\}$ indexed by elements of G. Let $U \subset V$ be a vector subspace spanned by vectors $e_{gh} - e_{hg}$ for any $g, h \in G$. Suppose that the quotient vector space V/U is finite-dimensional. Show that G has only finitely many conjugacy classes and that their number is equal to $\dim V/U$.

3. Let χ be a character of a complex representation of a finite group G. Show that the function

$$g \mapsto 2 + 3\chi(g)$$

is also a character of G.

2. Commutative Algebra

4. Let R be a commutative ring with unity and let $\mathfrak{p} \subset R$ be a prime ideal.

- **a.** Show that the localization $R_\mathfrak{p}$ is a field if and only if for any element $x \in \mathfrak{p}$ there exists $y \notin \mathfrak{p}$ such that $xy = 0$.
- **b.** Find an example of a commutative ring R and a prime ideal $\mathfrak{p} \neq 0$ such that $R_\mathfrak{p}$ is a field.

5. Let R be a domain and let $I \subset R$ be an ideal. An element $x \in R$ is called *integral over* I if it satisfies an equation of the form

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0$$

with $a_k \in I^k$, the k-th power of the ideal I, for each k.[1] Show that x is integral over I if and only if there exists a finitely generated R-module M, not annihilated by any element of R, such that $xM \subset IM$.

[1] Be careful: this notion of integrality over an ideal is different from (although related to) the notion of integrality over a ring.

6. Let R be a commutative ring with unity that contains only finitely many maximal ideals and such that for each maximal ideal $\mathfrak{m}$ of R, the localization $R_{\mathfrak{m}}$ is Noetherian. Prove that

a. The product of localization maps $R \to \oplus_{\mathfrak{m}} R_{\mathfrak{m}}$ is an embedding.
b. R is Noetherian.

3. Field Theory and Galois Theory

7. Find the minimal polynomial of $\sqrt{2} - \sqrt{3}$ over

a. $\mathbb{Q}$;
b. $\mathbb{Q}[\sqrt{3}]$.

8. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers. Let $\mathbb{Q}[i]$ be its quotient field. Let K be a finite Galois extension of $\mathbb{Q}[i]$. Let G be the Galois group of K over $\mathbb{Q}[i]$. Let $R \subset K$ be the integral closure of $\mathbb{Z}[i]$ in K. Show that $\sigma(R) \subset R$ for any $\sigma \in G$ and that $R^G = \mathbb{Z}[i]$.

9. Find the Galois group of the polynomial $X^3 - X - t$ over the field $\mathbb{C}(t)$.