Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each question is worth 20 points.

1. Suppose you are told to toss a die until you have observed each of the six faces.
   (a) Let \( Y_1 \) be the trial on which the first face is tossed, \( Y_2 \) be the number of additional tosses required to get a face different than the first, \( Y_3 \) be the number of additional tosses required to get a face different than the first two distinct faces, \ldots, and \( Y_6 \) be the number of additional tosses required to get the last remaining face after all other faces have been observed. Find the distribution of each \( Y_i \), \( i = 1, \ldots, 6 \).
   (b) What is the expected number of tosses required in order to observe each of the six faces?

2. (a) Suppose \( X \sim N(0, 1) \). Find the p.d.f. of \( Y = X^2 \).
   (b) Let \( X_1 \) and \( X_2 \) be two independent random variables; \( X_1 \) has an exponential distribution with mean 1, and \( X_2 \) has an exponential distribution with mean 2. Find the p.d.f. of \( Y = 2X_1 + X_2 \).
   (c) Let \( X_1 \) and \( X_2 \) be two independent exponentially distributed random variables, each with mean 1. Find \( P(X_1 > X_2 | X_1 < 2X_2) \).

3. Let \( Z \) be a standard normal random variable and let \( Y_1 = Z \) and \( Y_2 = Z^2 \).
   (a) Find \( E(Y_1) \), \( E(Y_2) \), and \( E(Y_1Y_2) \).
   (b) Find \( Cov(Y_1, Y_2) \). Are \( Y_1 \) and \( Y_2 \) independent?

4. Suppose that \( X_1, \ldots, X_k \) are iid \( N(\mu, \sigma^2) \), \( k \geq 2 \). Denote:
   \[
   U_1 = \sum_{i=1}^{k} X_i, U_j = X_1 - X_j \text{ for } j = 2, \ldots, k
   \]
   (a) Show that \( U = (U_1, \ldots, U_k) \) has a \( k \)-dimensional normal distribution;
   (b) Show that \( U_1 \) and \( (U_2, \ldots, U_k) \) are independent;
   (c) Express \( S^2 \) as a function of \( U_2, \ldots, U_k \) alone. Hence, show that \( \bar{X} \) and \( S^2 \) are independently distributed. (Hint: You may use the fact that \( \left( \frac{k}{2} \right) S^2 = \sum_{1 \leq i < j \leq k} \frac{1}{2} (X_i - X_j)^2 \)).
5. (a) Let \( \{\xi_n, n \geq 1\} \) be a sequence of independently identically distributed random variables with \( E(\xi_1) = \mu, \ Var(\xi_1) = \sigma^2 < \infty, \) and \( P(\xi_1 = 0) = 0. \) Prove that
\[
\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2} \to \frac{\mu}{\mu^2 + \sigma^2}, \quad n \to \infty,
\]
in probability. (Hint: you may use the theorem that says if \( X_n \) converges to \( X \) in probability and \( Y_n \) converges to \( Y \) in probability, and if \( f \) is continuous, then \( f(X_n, Y_n) \) converges to \( f(X, Y) \) in probability. If further \( X = a \) and \( Y = b \) are constants, then \( f \) only needs to be continuous at \((a, b)\).)

(b) Let
\[
X_n = \begin{cases} 
  n & \text{with probability } 1/n \\
  0 & \text{with probability } 1 - 1/n.
\end{cases}
\]
Show that \( X_n \) converges in probability to zero, but \( E(X_n) \) and \( Var(X_n) \) do not converge to zero.