DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. A complex-valued function $f(x)$ and its Fourier transform $g(k)$ are related by the following identities:

$$g(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \,,$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k) e^{ikx} dk \,.$$

Use the above relations to derive the Plancherel identity:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(k)|^2 \, dk \,.$$

2. The motion of a semi-infinite string is governed by the following set of equations:

$$\phi_{tt} - \phi_{xx} = 0, \quad t > 0, \ \ x > 0,$$

$$\phi(x,0) = \cos(x), \ \ \phi_t(x,0) = 0, \quad x > 0,$$

$$\phi(0,t) = e^{-t}, \quad t > 0 \,.$$

(a) Write down the general solution of the above partial differential equation (which contains two arbitrary functions, each being constant along a family of characteristics).

(b) Determine the solution of the above initial/boundary-value problem in the region $x > t$. [Note that the solution that you construct will only be valid when the arguments of the two arbitrary functions are positive.]

3. Solve the linear, first-order partial differential equation

$$u_t + (1 + x)u_x = 0, \quad x > 0, \quad t > 0.$$

with $u(x, 0) = f(x)$ and $u(0, t) = g(t)$.

4. Determine the stability of the critical point 0 of the van der Pol equation

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0,$$

for all values of the real parameter $\mu \in R$.

5. Consider the Hamiltonian system with one degree of freedom:

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q},$$

for a given smooth function $H = H(q, p)$.

(a) Show that a fixed/equilibrium point $(\bar{q}, \bar{p})$ cannot be asymptotically stable.

(b) Give an example of an H for which the system has both a stable and an unstable equilibrium point.

6. Consider the differential equation

$$\frac{d^2x}{dt^2} + \frac{1}{x^2} - \frac{1}{x^3} = 0 \quad \text{for} \quad x > 0\,.$$

(a) Sketch the phase portrait.

(b) Identify all equilibrium points/solutions and all periodic solutions.

7. Consider the Boundary Value Problem (BVP)

$$u_{xx} + u_{yy} = 0 \;\; \text{in} \quad x^2 + y^2 < a^2,$$

$$u(x, y) = f(x, y) \;\; \text{for} \quad x^2 + y^2 = a^2,$$

(a) Restate this BVP in polar coordinates: $x = r\cos\theta, y = r\sin\theta$.

(b) Solve it explicitly for the boundary data

$$f = \cos(2\theta) - \sin(3\theta)$$