Do 7 of the following 9 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. Denote by $X$ the set of six vectors $(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1)$.

Find two different, non-empty subsets $Y_1, Y_2$ of $X$ such that

- the elements of each $Y_i$ are linearly independent, and
- the elements of $Y_i \cup \{\vec{x}\}$ are not linearly independent for any $\vec{x} \in X \setminus Y_i$.

Justify your answer!

2. Let $\vec{w} \in \mathbb{R}^n$ be a unit vector. Define a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as follows:

$$T\vec{x} := \vec{x} - 2(\vec{x} \cdot \vec{w})\vec{w}$$

(Where $\vec{x} \cdot \vec{w}$ is the usual inner product in $\mathbb{R}^n$).

(a) Show that $T$ is an orthogonal transformation, in other words $||T\vec{x}|| = ||\vec{x}||$ for all $\vec{x}$.

Hint: What is the geometric interpretation of $T$? You might want to draw a picture.

(b) Find the Jordan form of $A$.

3(a) Let $A, B$ be $n \times n$ matrices. If $AB = 0$, show that

$$\text{rank}(A) + \text{rank}(B) \leq n.$$ 

(b) For any $n \times n$ matrix $A$, show that there exists a $n \times n$ real matrix $B$ with

$$AB = 0 \quad \text{and} \quad \text{rank}(A) + \text{rank}(B) = n.$$ 

4. Suppose $A$ is a real $n \times n$ matrix with all entries $\geq 0$ and with the sum of entries in each column equal to 1.

(a) Show that $A$ has an eigenvector with eigenvalue equal to 1.

(b) Show that all eigenvalues $\lambda$ of $A$ satisfy $|\lambda| \leq 1$

Hint: One way to do this is prove the corresponding statement for $A'$; of course there are other ways.
Part II. Advanced Calculus

1. The *Fundamental Theorem of Arithmetic* says that every integer \( n > 1 \) can be written uniquely as

\[
n = p_1^{e_1} \cdots p_r^{e_r},
\]

where \( p_1 < \cdots < p_r \) are primes and the \( e_i \) are positive integers. Use the Fundamental Theorem to show that if \( \{n_i\}_{i \in \mathbb{N}} \) is an infinite, strictly increasing sequence of positive integers such that the series \( \sum_{i=1}^{\infty} 1/n_i \) diverges, then the set

\[
\{ p \text{ prime : } p \text{ divides } n_i \text{ for some } i \}
\]

is infinite.

2. Fix numbers \( R > r > 0 \). Compute the volume of the solid obtained by rotating the circle \((x - R)^2 + y^2 = r^2\) above the \( y \)-axis. Show your work.

3. Let \( f_1(x, y), f_2(x, y) \) be smooth functions on \( \mathbb{R}^2 \). Denote by \( X_i \) the surface in \( \mathbb{R}^3 \) defined by \( z = f_i(x, y) \). Suppose \( X_1 \cap X_2 = \emptyset \). As \( p_i \) runs through all points on \( X_i \), show that the line segment \( \overline{p_1p_2} \) is perpendicular to both \( X_i \) whenever the length of the line segment reaches a local minimum or local maximum.

4. Let \( f : [0, 1] \to \mathbb{R} \) be a Riemann integrable function. It is a fact that for any integer \( n > 0 \), the function \( g_n(x) := f(x^n) \) is also Riemann integrable on \( [0, 1] \).

(a) If \( f \) is continuous at \( x = 0 \), show that

\[
\lim_{n \to \infty} \int_0^1 g_n(t)dt = f(0).
\]

(b) Give an example to show that (1) is false if \( f \) is not continuous at \( x = 0 \).

5. Let \( f(x, y) = xy + \int_0^y \sin(t^2)dt \).

(a) Compute \( \nabla f(a, b) \).

(b) Show that \( (0, 0) \) is a saddle point of \( f(x, y) \).