DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM - DIFFERENTIAL EQUATIONS
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Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1. Suppose that the 2-dimensional linear system $x' = Ax$ has a general solution of the form

$$x(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t}.$$

   (a) Find the matrix A for this system.
   (b) Find the exponential matrix e^{tA}.

2. Consider the 3-dimensional system of ODE's,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} y \\ z \\ 1 - x^2 + y \end{pmatrix}$$

   (a) Describe the behavior of solutions in small neighborhoods of the two rest points $(\pm 1, 0, 0)$.
   (b) Let $V(x, y, z) = yz - x + x^3/3$, and let $v(t) = V(x(t), y(t), z(t))$, where $(x(t), y(t), z(t))$ is a solution of the ODE's. Suppose that $v(t) = c$ for some constant c for all real t. Show that the solution must be one of the two rest points.
   (c) Suppose that there is a nonconstant solution $(x(t), y(t), z(t))$ of the ODE's such that

$$|x(t)| + |y(t)| + |z(t)| \leq M$$

   for all t and for some constant $M > 0$. Show that

$$\lim_{t \to -\infty} (x(t), y(t), z(t)) = (1, 0, 0), \quad \lim_{t \to \infty} (x(t), y(t), z(t)) = (-1, 0, 0).$$

   (It is sufficient to give a detailed analysis for one of the two limiting directions.)

3. Consider the system

$$\begin{aligned} x' &= 2 - x^2 - y \\ y' &= -y - x. \end{aligned} \tag{1}$$

(a) Use linearization to analyze the stability and local behavior of various solutions in the stable and/or unstable manifolds of the two critical points (-1,1) and (2,-2) of (**??**) in small neighborhoods of each rest point.

(b) Sketch the null sets $x' = 0$ and $y' = 0$ for the system (**??**), and use the analysis in (a) to sketch the behavior of the solutions in the stable and/or unstable manifolds of (-1,1) and (2,-2) in small neighborhoods of the two rest points. Your figure should depict the behavior of solutions in these manifolds with respect to the the null sets $x' = 0$ and $y' = 0$. Justify your drawing with appropriate analytical calculations based on information about the linearized systems.

(c) Give a rigorous proof of the existence of a solution $(x(t), y(t))$ of (**??**) running from (-1,1) as $t \to -\infty$ to (2,-2) as $t \to +\infty$.

4. Solve the Cauchy problem

$$(x + y)\, u_x + y\, u_y = 1,$$
$$u(x, 1) = x, \quad \text{for} \quad 0 < x < 1.$$

Describe the region over which the solution is uniquely determined.

5. Find Green's function for Laplace's equation $-\Delta u = 0$ in the region

$$U = \{x \in R^3 : x_1 > 0, x_2 > 0\}.$$

[Hint: first consider a half-plane.]

6. Consider the equation

$$u_t = \epsilon\, u_{xx}$$

with periodic boundary conditions,

$$u(-L,t) = u(L,t), \qquad u_x(-L,t) = u_x(L,t),$$

which models heat flow in a circular ring of circumference $2L$.

(a) Find *all* eigenfunctions of this BVP.

(b) Show that the initial boundary value problem has a solution for any periodic initial data $u_0 \in L^1 \cap L^2$.

(c) Show that this solution is a *classical* solution in the domain $t > 0$.

7. Consider the IBVP for the wave equation with Robin boundary conditions

$$\begin{aligned} u_{tt} &= \tau\, u_{xx}, \qquad (0 < x < L,\ t > 0) \\ u(x,0) &= \phi(x), \qquad u_t(x,0) = \psi(x) \\ k_0\, u(0,t) - \tau\, u_x(0,t) = 0,&\quad k_1\, u(L,t) + \tau\, u_x(L,t) = 0, \end{aligned}$$

with τ, k_0 and k_1 positive constants. Show that the energy

$$E(t) = \frac{1}{2}\int_0^L u_t^2\, dx + \frac{1}{2}\int_0^L \tau\, u_x^2\, dx + \frac{1}{2}k_0\, u(0,t)^2 + \frac{1}{2}k_1\, u(L,t)^2$$

is constant, and use this result to show that there can be at most one solution to the initial-boundary value problem.