Department of Mathematics and Statistics
University of Massachusetts Amherst

Advanced Exam – Algebra
Fall 2010

Passing Standard: It is sufficient to do five problems correctly, including at least one from each of the three parts.

1. Group Theory and Representation Theory

1. Let $G$ be a finite group and let $V$ be an irreducible complex representation of $G$.
   
   (a) Let $x \in V$, $x \neq 0$. Prove that $\dim V \leq [G : G_x]$. (Here $G_x$ is the stabilizer of $x$ for the action of $G$ on $V$.)
   
   (b) Let $H \subset G$ be an Abelian subgroup. Prove that $\dim V \leq [G : H]$.

2. Let $p$ be a prime number and let $G$ be a finite $p$-group. Let $H \subset G$ be a proper subgroup. Prove that the normalizer of $H$ in $G$ is larger than $H$:

   $$N_G(H) \neq H.$$  

3. Let $X$ be a set with at least two points. Let $G$ be a group acting doubly transitively on a set $X$: that is, for any $x_1, x_2 \in X$ and $y_1, y_2 \in X$ such that $x_1 \neq x_2$ and $y_1 \neq y_2$, there is a $g \in G$ such that $gx_1 = y_1$ and $gx_2 = y_2$. Show that for any $x \in X$, the stabilizer $G_x$ is a maximal proper subgroup of $G$. (That is, $G_x \neq G$ and there are no proper subgroups $H$ of $G$ such that $G_x \subset H$.)
2. Commutative Algebra

4. Let $R$ be a commutative ring and let $M$ be an $R$-module. Recall that $M$ is called flat if, for any short exact sequence of $R$-modules
\[ 0 \to N' \to N \to N'' \to 0, \]
the induced sequence
\[ 0 \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0 \]
is also exact.

(a) Let $M$ be a flat $R$-module, let $r \in R$ be a non-zero-divisor, and let $m \in M$ be such that $rm = 0$. Prove that $m = 0$.

(b) Prove that an $R$-module $M$ is flat if and only if the localization $M_p$ is a flat $R_p$-module for any prime ideal $p \subset R$.

5. Let $R$ be a commutative domain. Show that if $R[x]$ is a principal ideal domain, then $R$ is a field.

6. Let $R$ denote a commutative ring containing a field $F$. Suppose that $R$ is finite dimensional as an $F$-vector space.

(a) Prove that any prime ideal of $R$ is maximal.

(b) Prove that $R$ has finitely many maximal ideals.

3. Galois Theory

7. Let $K$ be a field and let $G$ be a finite group of automorphisms of $K$. Let $H \subset G$ be a subgroup. Prove that there exists $x \in K$ such that
\[ H = \{ g \in G \mid g \cdot x = x \}. \]

8. Let $p$ be a prime number and let $n$ be a positive integer. Prove that $GL_n(F_p)$ contains an element of order $p^n - 1$.

9. Let $K$ be a field containing a cube root of unity $\omega$ and let $L/K$ be a Galois extension with Galois group cyclic of order 3.

(a) Prove that there is $\beta \in L$ such that $\sigma(\beta) = \omega \beta$, where $\sigma$ is a generator of Gal($L/K$).

(b) Prove that there is $\alpha \in K$ such that $L = K(\sqrt[3]{\alpha})$. 