Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master’s level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

In the following, $C(X,Y)$ denotes the set of continuous functions from topological spaces $X$ to $Y$, and $\mathbb{R}$ denotes the real line with the standard topology.

(1) Let $T$ be the family of subsets of $\mathbb{R}^2$ consisting of the empty set and the complements of finite unions of points and lines. Show that $T$ is a topology. Is $(X,T)$ Hausdorff?

(2) Let $X$ and $Y$ be two locally compact Hausdorff topological spaces.

(a) Define the one-point compactification of $X$.

(b) Recall that a function is proper if the inverse image of any compact set is compact. Prove that a function $f: X \rightarrow Y$ is proper if and only if it extends to a continuous map between the one-point compactifications of $X$ and $Y$.

(3) Let $\{X_i \mid i \in I\}$ be a collection of topological spaces, and for each $i \in I$ let $A_i \subset X_i$. Show that $\prod A_i$ is dense in $\prod X_i$ if and only if each $A_i$ is dense in $X_i$.

(4) Let $X$ be a set, and let $f_n: X \rightarrow \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Prove that $\{f_n\}$ converges uniformly to a function $f: X \rightarrow \mathbb{R}$ if and only if $\{f_n\}$ converges in $(\mathbb{R}^X, \bar{\rho})$.

(5) Let $(X,d)$ be a metric space and $f: X \rightarrow X$ a function such that $d(x,y) > d(f(x), f(y))$ for all $x \neq y$ in $X$.

(a) Give an example to show that if $X$ is not compact, then $f$ need not have any fixed points.

(b) If $X$ is compact, show that $f$ has a fixed point $x_0$.

(c) Show that if $f$ has a fixed point, it is unique.

(6) Let $f: X \rightarrow Y$ be a continuous bijection.

(a) Show that if $X$ is compact and $Y$ is Hausdorff, then $f$ is a homeomorphism.

(b) Give an example where $Y$ is Hausdorff and $f$ is not a homeomorphism.

(c) Give an example where $X$ is compact and $f$ is not a homeomorphism.

(7) If $A \subseteq X$, a retraction of $X$ onto $A$ is a continuous map $r: X \rightarrow A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.