Department of Mathematics and Statistics
Basic Probability Exam
August 2009

Work all problems. Show your work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters level and 75 to pass at the Ph.D. level

1. An urn contains nine chips, five red and four white. Three are drawn out at random without replacement. let X denote the number of red chips in the sample. Let Y denote the payment in dollars received by player, depending on X such that $Y = (-2)^X$.

   (a) **(6 pt)** Find the distribution of X.

   (b) **(4 pt)** Compute the expected payment of a player, i.e., that of Y.

2. Consider a random variable Y with probability density function (pdf) given by

$$f(y) = ce^{-y^2/2}, \quad -\infty < y < \infty$$

   (a) **(5 pt)** Find c.

   (b) **(5 pt)** Derive the moment-generating-function of Y.

   (c) **(5 pt)** Find the expected value and variance of Y.

   (d) **(10 pt)** What is the pdf of Y^2?

3. Let X_1 and X_2 be independent standard normal random variables. Let U be independent of X_1 and X_2, and assume that U is uniformly distributed over (0,1). Define $Z = UX_1 + (1-U)X_2$.

   (a) **(5 pt)** Find the conditional distribution of Z given $U = u$

   (b) **(5 pt)** Find the expected value of Z, E(Z)

   (c) **(15 pt)** Find the variance of Z, V(Z)

4. **(15 pt)** A blood test is 99 percent effective in detecting a certain disease when the disease is present. However, the test also yields a false-positive result for 2 percent of the healthy patients tested, who have no such disease. Suppose 0.5 percent of the population has the disease. Find the conditional probability that a randomly tested individual actually has the disease given that his or her test result is positive.

5. (a) **(5 pt)** State carefully the Central Limit Theorem for a sequence of i.i.d. random variables.

   (b) **(5 pt)** Suppose $X_1, \ldots, X_{100} \sim$ i.i.d. Unif$(0,1)$. What is the standard deviation of $\bar{X}$, the mean of $X_1, \ldots, X_{100}$?

   (c) **(15 pt)** Use the Central Limit Theorem to find approximately the probability that the average of the 100 numbers chosen exceeds 0.56. You may use the approximation $1/\sqrt{12} \approx 0.3$.