Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1. Consider the linear ODE
\[ 2x \frac{dy}{dx} + y = f(x), \quad \text{for } x \geq 0, \]
in which \( f \) is a smooth and strictly negative function. Show that it is not possible for any solution \( y(x) \) to have a finite and positive initial value \( y(0) \). How do the positive solutions behave as \( x \) approaches 0?

2. A nonlinear oscillator with displacement \( x \in \mathbb{R}^1 \) is governed by the DE:
\[ \frac{d^2x}{dt^2} + \frac{dV}{dx} = 0, \]
with potential \( V = \frac{1}{2}x^2 - \frac{1}{3}x^3 \).

   (a) Reformulate this dynamical equation as a two-dimensional system of first-order equations. Determine the equilibrium points of the system.

   (b) Analyze the stability of each equilibrium point and sketch the entire phase portrait.

   (c) Find a function \( H = H(x, \dot{x}) \) on the phase plane that is constant on each solution trajectory.
3. Consider the competing species model governing the evolution of two ecological species quantified by $x_1$ and $x_2$ (with $x_1, x_2 \geq 0$):

\[
\begin{align*}
\frac{dx_1}{dt} &= r_1 \left(1 - \frac{x_1}{k_1}\right)x_1 - c_1 x_1 x_2, \\
\frac{dx_2}{dt} &= r_2 \left(1 - \frac{x_2}{k_2}\right)x_2 - c_2 x_1 x_2.
\end{align*}
\]

(a) Interpret the positive constants $r_i$, $k_i$, and $c_i$ ($i = 1, 2$), and describe the meaning of the model’s terms that are scaled by these constants.

(b) In the case of “weak competition” when $c_1 < r_1/k_2$ and $c_2 < r_2/k_1$, determine the qualitative behavior of solutions as $t \to +\infty$? What does this behavior mean in ecological terms? Do find the nullclines and equilibrium point(s) and sketch them, but do not carry out a full stability analysis of the equilibrium point(s) [because the algebra is too messy.]

4. (a) Provide a complete derivation of the Laplace operator in polar coordinates in $\mathbb{R}^2$. That is, show that the operator $u \mapsto \triangle u = \partial^2 u / \partial x^2 + \partial^2 u / \partial y^2$ converts to

\[
\triangle u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.
\]

(b) Solve the following boundary value problem:

\[
\begin{align*}
\triangle u &= 0 \quad \text{in} \quad 0 \leq r < a, \ 0 \leq \theta < 2\pi, \\
u &= \sin^2 \theta \quad \text{on} \quad r = a.
\end{align*}
\]

5. The probability density function (PDF) $u(x, t)$ for an elastically bound particle evolves according to the equation

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial (xu)}{\partial x},
\]

for $-\infty < x < \infty$ and $t > 0$, where $D$ and $\gamma$ are positive constants. Verify that for all $t > 0$, the solution $u(x, t)$ is a PDF provided the data $u(x, 0)$ is. A function $v(x)$ is a PDF if and only if it satisfies both conditions

\[
v(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} v(x) \, dx = 1.
\]
6. The “shallow water equations” approximate the motion of a thin layer of incompressible and inviscid fluid in the presence of gravity. In one space dimension they are the following pair of nonlinear PDEs:

\[
\begin{align*}
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} &= -h \frac{\partial u}{\partial x}, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} &= -g \frac{\partial h}{\partial x}.
\end{align*}
\]

for a pair of unknowns, \( h = h(x,t), \ u = u(x,t) \) that represent the water surface height and the fluid velocity (in the \( x \) direction). The constant \( g \) is the gravitational acceleration.

(a) Consider motions that are small perturbations around the uniform, undisturbed state \( h = H, u = 0 \), where \( H \) is a constant water height. In terms of the perturbation variables \( \eta = h - H \) and \( u \), derive the linearized equations of motion (in which terms of higher order than the first in the perturbations are neglected).

(b) Show that this pair of linear first-order PDEs in the variables \( \eta = h - H \) and \( u \) are equivalent to a single, second-order PDE in \( \eta \); namely, the wave equation

\[
\frac{\partial^2 \eta}{\partial t^2} - c^2 \frac{\partial^2 \eta}{\partial x^2} = 0.
\]

Give a formula for the wave speed \( c \) in terms of \( g \) and \( H \).

7. The viscous Burgers’ equation for \( u(x,t) \),

\[
u_t + \left( \frac{1}{2} u^2 \right)_x = \epsilon \ u_{xx}, \quad (x \in \mathbb{R}^1, \ t > 0)
\]

is a fundamental equation for nonlinear viscous flows.

(a) Make the substitution

\[
w(x,t) = \int_{-\infty}^{x} u(\xi, t) \ d\xi,
\]

and derive the PDE satisfied by \( w(x,t) \).

(b) Now make the substitution

\[
w(x,t) = \alpha \log \phi(x,t), \quad \text{where } \alpha \text{ is a positive constant},
\]

and derive an equivalent PDE for \( \phi \).

(c) Conclude that for an appropriate choice of constant \( \alpha \), solutions \( u \) of Burgers’ equation are in 1-1 correspondence with solutions \( \phi \) of the heat equation. This is known as the “Cole-Hopf transformation.”