University of Massachusetts  
Department of Mathematics and Statistics  
Advanced Exam in Geometry  
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Do 5 out of the following 7 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

Problem 1. Prove the following version of the Frobenius Theorem: Let $M$ be a 2-manifold and $X,Y$ be linearly independent, non-vanishing smooth vector fields defined in a neighborhood of a point $p \in M$ such that $[X,Y] \equiv 0$. Show that there is a smooth chart $(x_1,x_2)$ centered at $p$ such that $X = \frac{\partial}{\partial x_1}$ and $Y = \frac{\partial}{\partial x_2}$.

Problem 2. Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology groups of $\mathbb{C}P^n$. In this problem you may assume the following knowledge about the de Rham cohomology groups of $S^n$: $H^k_{\text{dR}}(S^n) = \mathbb{R}$ if $k = 0,n$, $H^k_{\text{dR}}(S^n) = 0$ otherwise, and the fact that $\mathbb{C}P^1 = S^2$.

Problem 3. Let $M$ be a simply-connected, closed smooth 4-manifold, and let $\beta$ be a closed 3-form on $M$. Show that there exists a 2-form $\alpha$ on $M$ such that $\beta = d\alpha$.

Problem 4. Let $M$ be a closed, oriented, smooth $n$-manifold, and $f : M \to M$ be a smooth self-map. A fixed point $x \in M$ of $f$ (i.e., $f(x) = x$) is called simple if the map $f_* : T_x M \to T_x M$ is an isomorphism.

1) Let $\Delta := \{(x,x) | x \in M\} \subset M \times M$ be the diagonal and $\Gamma_f := \{(x,f(x)) | x \in M\} \subset M \times M$ be the graph of $f$. Show that the submanifolds $\Delta$ and $\Gamma_f$ intersect transversely in $M \times M$ if and only if $f$ has only simple fixed points.

2) Suppose $f$ has only simple fixed points. Explain how to orient $M \times M$, $\Delta$ and $\Gamma_f$ canonically so that the intersection number $\#(\Gamma_f, \Delta) = \sum_{x \text{ is a fixed point of } f} \text{sign(det}(f_* - \text{Id}))$.

Problem 5. Let $Q$ be the diagonal matrix with entries 1 and $-1$, and define $G \subset SL(2,\mathbb{C})$ to be the set of all 2-by-2 complex matrices $A$ with determinant 1 such that $A^* QA = Q$, where $A^*$ is the conjugate transpose of $A$.

a) Prove that $G$ is a Lie group, calculate its Lie algebra and compute its dimension.

b) Determine a maximal abelian subalgebra of its Lie algebra and a corresponding maximal torus in $G$. 


Problem 6. Let $M$ be a smooth $n$-manifold and let $V(M)$ be the associated bundle of $n$-frames.

a) Show that $V(M)$ is a smooth principal $GL_n(\mathbb{R})$-bundle over $M$.

b) Show that $V(M)$ is a trivial bundle if and only if $TM$ is a trivial bundle.

Problem 7. Let $(M, \langle \cdot, \cdot \rangle)$ be a 2-dimensional Riemannian manifold, and let $\nabla$ be the Levi-Civita connection. For any point $x \in M$, define

$$K(x) \equiv \frac{\langle R(X,Y)Y, X \rangle}{\sqrt{|X|^2|Y|^2 - \langle X, Y \rangle^2}},$$

where $X,Y \in T_xM$ is a pair of linearly independent vectors. (Here $R(X,Y)Z \equiv \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ is the curvature endomorphism.) Show that

1. $K(x)$ depends only on $x$ (i.e., independent of the choice of $X,Y$).
2. If $K \equiv 0$ on $M$, then $\langle \cdot, \cdot \rangle$ is locally isometric to the Euclidean metric.