Do five of the following problems. All problems carry equal weight.

Passing level:
Masters: 60% with at least two substantially correct.
Ph.D.: 75% with at least three substantially correct.

1. Find the Padé approximation

\[ R_{1,1}(x) = \frac{a + bx}{1 + cx} = p_2(x) + O(x^3) \]

of the polynomial \( p_2(x) = 1 - \frac{1}{2} x + \frac{1}{24} x^2 \), and use it to deduce the approximation

\[ \cos(x) = \frac{12 - 5x^2}{12 + x^2} + O(x^6). \]

2. Consider the fixed point iteration defined by the formula \( x_{n+1} = F(x_n) \), where

\[ F(x) = x - a + 2ae^{-x}. \]

Here \( a \neq 0 \) is a parameter.

(a) Find the fixed point, \( p \).

(b) Determine for which values of \( a \) the iteration converges to \( p \) and those for which they diverge away from \( p \). Your answer must be in terms of intervals or unions of intervals.

(c) Does there exist a value of \( a \) for which the iteration converges quadratically? If so, find it.
3. Consider the numerical integration rule

\[ I(f) = \int_{-h}^{h} f(x) \, dx \approx A_0 f(-h/2) + A_1 f(0) + A_2 f(h/2). \]

(a) Find \( A_0, A_1, \) and \( A_2 \) such that the integration rule is exact for polynomials of degree \( \leq 2. \)

(b) Show that the rule constructed in (a) is in fact exact for polynomials of degree \( \leq 3. \)

(c) For the constructed rule, it can be proved that

\[ I(f) - [A_0 f(-h/2) + A_1 f(0 + A_2 f(h/2))] = c_0 f^{(4)}(\eta)h^5, \quad \eta \in (-h, h) \]

where \( c_0 \) is a constant independent of \( f. \) Find the constant \( c_0. \)

4. Consider the ODE system

\[ u_t = -Au, \]

where \( A \) is a constant symmetric positive definite matrix.

(a) Construct a fourth-order numerical scheme for the above system.

(b) Give the stability condition for this scheme.

(Hint: First consider a similarity transformation of \( A. \))

5. An \( n \times n \) matrix of the form \( N(y, k) = I - ye_k^T \) is called a Gauss-Jordan matrix. Here \( e_k \) is the \( k\)th unit coordinate vector, and \( y \) is an arbitrary vector.

(a) Find a formula for \( N(y, k)^{-1} \). Under what conditions does this inverse exist?

(b) Let \( x \) be an arbitrary vector. For a given \( k \), find a vector \( y \) so that \( N(y, k)x = e_k \). Under what conditions does such a \( y \) exist?

(c) For a given square matrix \( A \), give an algorithm based on Gauss-Jordan matrices that computes \( A^{-1} \). Under what conditions will this algorithm work?
6. Consider the difference scheme

\[ y_{n+1} = \alpha y_{n-1} + \beta y_n + \gamma h f(x_n, y_n) \]

for approximating the solution to the equation

\[ \frac{dy}{dx} = f(x, y(x)). \]

Find constants \( \alpha, \beta \) and \( \gamma \) for which this has highest order, and the corresponding local truncation error.

7. Given a function \( f(x) \), and a set of Gauss quadrature points and weights: \( \{x_i\}_{i=1}^n, \{\omega_i\}_{i=1}^n \). Let \( P_k(x) \) be the Legendre polynomial of degree \( k \).

(a) Define the interpolation of \( f(x) \) on the space \( \text{span}\{P_0(x), P_1(x), \ldots, P_{n-1}(x)\} \), with interpolation points \( \{x_i\}_{i=1}^n \)

(b) Find the \( L^2 \) projection of \( f(x) \) onto the above space while using the given Gauss quadrature to evaluate the integrals.

(c) Are the two approximate functions obtained in (a) and (b) the same? Give a detailed explanation.