BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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Do 7 of the following 9 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1(a) Find an *orthonormal basis* of the subspace V of $\mathbf{R}^4$ spanned by the column vectors

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

(b) Let $P : V \to V$ be the orthogonal projection of V onto the plane spanned by $\vec{x}_1$ and $\vec{x}_2$. Calculate the matrix of P with respect to the *ordered basis* $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ of V.

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2. Denote by $M_n(\mathbf{C})$ the complex vector space of all $n \times n$ complex matrices. Fix $A \in M_n(\mathbf{C})$, and denote by $T_A : M_n(\mathbf{C}) \to M_n(\mathbf{C})$ the linear transformation given by

$$T_A(X) = AX \qquad (X \in M_n(\mathbf{C})).$$

(a) Show that T_A and A have the same eigenvalues.

(b) Express the characteristic polynomial of T_A in terms of the characteristic polynomial of A.

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3. Let V be an n-dimensional complex vector space and $T : V \to V$ a linear transformation. Let $v_1, \ldots, v_n$ be non-zero vectors in V, each an eigenvector for T corresponding to a different eigenvalue. Show that $v_1, \ldots, v_n$ are linearly independent.

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4. Let

$$A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{pmatrix}.$$

(a) Determine conditions on a, b, c, d so that there is only one Jordan block for each eigenvalue of A in the Jordan form of A.

(b) Find the Jordan form of A when $a = c = d = 2$ and $b = -2$.

Part II. Advanced Calculus

1. Let f be a continuous function on $[a, b]$. If

$$\int_a^b f(x)g(x)dx = 0$$

for every continuous function g on $[a, b]$, show that f is identically zero on $[a, b]$.

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2. Let $\{f_n(x)\}_n$ be a sequence of functions on $\mathbf{R}$ such that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly and (pointwise) absolutely on $\mathbf{R}$. Does that mean $\sum_{n=1}^{\infty} |f_n(x)|$ converges uniformly on $\mathbf{R}$? Prove or give a counterexample.

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3. If a sequence $\{a_n\}_n$ of positive numbers converges to a finite number A, show that

$$\lim_{n\to\infty} \sqrt[n]{a_1 \cdots a_n} = A.$$

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4. Let $f, g : \mathbf{R}^3 \to \mathbf{R}$ be differentiable functions. Suppose the gradient of f satisfies

$$\nabla f(\vec{x}) = g(\vec{x})\vec{x} \qquad \text{for every } \vec{x} \in \mathbf{R}^3.$$

Show that f is constant on every sphere centered at the origin.

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5. Let K be the the solid 'upside-down' right circular cone whose vertex is at the origin and whose base is the unit circle in the plane $z = 1$ having its center on the z-axis. Let S be the entire surface of D, including the base, and orient S by its outward unit normals. Evaluate the surface integral

$$\iint_S \vec{F} \cdot dS$$

where $\vec{F} = x^2\,\vec{i} + y\,\vec{j} + z\,\vec{k}$.
