UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
ADVANCED EXAM - "Mathematical Statistics" and Probability
Monday, August 25, 2008

Work all problems. 70 points are required to pass with at least 25 from each part ("Linear models/Mulivariate = questions 1 and 2; Probability = questions 3-5).

1. (30 PTS) Suppose that $\mathbf{X}$ $(p \times 1)$ has a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, (written $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$) where $\boldsymbol{\Sigma}$ is assumed to be non-singular.

   (a) Write down the density of $\mathbf{X}$ and then prove that the moment generating function of $\mathbf{X}$ is
   $$M(\mathbf{t}) = e^{\mathbf{t}'\mathbf{m} + (\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}/2)}.$$

   (b) Prove using moment generating functions that $\boldsymbol{a} + \mathbf{BX}$ is distributed as a q dimensional multivariate normal distribution with mean $\boldsymbol{a} + \mathbf{B}\boldsymbol{\mu}$ and covariance matrix $\mathbf{B}\Sigma\mathbf{B}'$, where $\boldsymbol{a}$ is a $q \times 1$ vector of constants and $\mathbf{B}$ a $q \times p$ matrix of constants.

   (c) Show how you can linearly transform $\mathbf{X}$ to $\mathbf{Z}$ such that $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$, where $\mathbf{Z}$ is $p \times 1$, $\mathbf{0}$ is a $p \times 1$ vector of 0's and $\mathbf{I}$ is the $p \times p$ identity matrix.

   (d) Let $\mathbf{A}$ be a $p \times p$ matrix and define $Q = \mathbf{X}'\mathbf{AX}$. Find $E(Q)$, expressing your answer in the simplest way you can as a a function of $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ and $\mathbf{A}$. Note: Do NOT use the normality of $\mathbf{X}$ here, but just the fact that $E(\mathbf{X}) = \boldsymbol{\mu}$ and $Cov(\mathbf{X}) = \boldsymbol{\Sigma}$.

   (e) State the definition of a non-central chi-square distribution with d degrees of freedom and non-centrality parameter λ. Do this not by giving a density function but by explaining how the distribution arises as the distribution of a random variable C, formed as a function of a suitably defined normal random vector.

   (f) Now return to the assumption that $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$ and state a necessary and sufficient condition on the matrix $\mathbf{A}$ such that $\mathbf{X}'\mathbf{AX}$ is distributed as a non-central chi-square.

   - Give the degrees of freedom and non-centrality parameter involved.

   - Prove the sufficiency.

2. (20 PTS) Consider random vectors $\mathbf{Y}$, $\mathbf{X}$ and $\mathbf{W}$. For convenience assume that all are continuous with density functions (so you can use integrals for expectations).

   (a) Prove that
   i) $E(\mathbf{Y}) = E_{\mathbf{X}}[E(\mathbf{Y}|\mathbf{X})]$
   ii) $Cov(\mathbf{Y}, \mathbf{W}) = Cov_{\mathbf{X}}[E(\mathbf{Y}|\mathbf{X}), E(\mathbf{W}|\mathbf{X})] + E_{\mathbf{X}}[Cov(\mathbf{Y}, \mathbf{W}|\mathbf{X})]$.
   Two notes. First, The outside expectation is written $E_{\mathbf{X}}$ to remind you that it is with respect to random $\mathbf{X}$. Second, there is nothing that says that $\mathbf{Y}$ can't be $\mathbf{W}$ so a special case of ii) is $Cov(\mathbf{Y}) = Cov_{\mathbf{X}}[E(\mathbf{Y}|\mathbf{X})] + E_{\mathbf{X}}[Cov(\mathbf{Y}|\mathbf{X})]$.

   (b) Now suppose that Y is a random variable with (unconditional) mean μ_Y and variance σ_Y^2, $\mathbf{X}$ is $p \times 1$ with $E(\mathbf{X}) = \boldsymbol{\mu}_X$, $Cov(\mathbf{X}) = \boldsymbol{\Sigma}_X$ (assumed non-singular) and $Cov(\mathbf{X}, Y) = \boldsymbol{\sigma}_{XY}$ ($p \times 1$). It is assumed that

   $$E(Y|\mathbf{X} = \mathbf{x}) = \beta_0 + \beta_1'\mathbf{x}$$

   and $V(Y|\mathbf{X} = \mathbf{x}) = \sigma^2$; this is our usual linear regression model. Use the previous part to PROVE that this assumption implies the following relationships:
   i) $\boldsymbol{\beta}_1 = \boldsymbol{\Sigma}_X^{-1}\boldsymbol{\sigma}_{XY}$
   ii) $\beta_0 = \mu_Y - \boldsymbol{\beta}_1'\boldsymbol{\mu}_X$
   iii) $\sigma^2 = \sigma_Y^2 - \boldsymbol{\beta}_1'\boldsymbol{\Sigma}_X\boldsymbol{\beta}_1$.
   Note: There is no normality assumption here, just assumptions on the moments.

3. (15 PTS) Let $\{X_n\}$ be a sequence of random variables, and suppose for each n that m_n is a median of X_n, i.e., that

   $$P(X_n \geq m_n) \geq \frac{1}{2} \leq P(X_n \leq m_n)\,.$$

   Show that if $X_n \to 0$ in probability (in measure), then $m_n \to 0$, though $E(X_n)$ need not converge to zero. Show the latter with a counterexample.

4. (20 PTS) Let $F(x)$ be a distribution function.

   (a) Show that

   $$\int_{-\infty}^{\infty} [F(x+c) - F(x)]dx = c\,,$$

   for every real c.

   (b) Show that if X is a positive random variable then

   $$E(X) = \int_{x=0}^{\infty} (1 - F(x))dx.$$

5. (15 PTS) Show that if ϕ is a real-valued characteristic function, then for all real t,

$$2\phi(t)^2 \leq 1 + \phi(2t)\,.$$

Hint: Recall that $2\cos^2\theta - 1 = \cos 2\theta$.