Do 5 out of the following 7 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

1. a) Show that
\[ M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 x_2 + x_3 x_4 = 0, \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \} \]
is a smooth submanifold of \( \mathbb{R}^4 \).

b) Show that \( M \) is orientable.

c) Find all critical points of the restriction of the coordinate function \( x_1 \) to \( M \).

2. Let \( M, N \) be smooth manifolds and \( F: M \to N \) a submersion. Let \( \Delta \) be a smooth \( k \)-dimensional distribution in \( N \) and define:
\[ F^*(\Delta)_p := (F_{*p})^{-1}(\Delta_{F(p)}) \subset T_p(M). \]
a) Prove that \( F^*(\Delta) \) is a smooth distribution in \( M \). What is the dimension of \( F^*(\Delta) \)?

b) Prove that \( F^*(\Delta) \) is integrable (involutive) if and only if \( \Delta \) is integrable (involutive).

c) Give an example to show that a) may fail if \( F \) is not a submersion.

3. A \( 2n \)-dimensional manifold is called symplectic if there exists a closed 2-form \( \omega \) for which the \( n \)th exterior power \( \omega \wedge \cdots \wedge \omega \) does not vanish at any point. Prove that

a) if \( M \) is symplectic with symplectic form \( \omega \), then \( X \mapsto \omega(X, -) \) defines an isomorphism of vector bundles \( TM \to T^*M \).

b) Any orientable, 2-dimensional manifold is symplectic.

c) For any \( n \geq 2 \), the sphere \( S^{2n} \) is not symplectic but the torus \( T^{2n} \) is.

4. Let \( D \) be the unit disc in \( \mathbb{R}^2 \) with the Euclidean metric, and let \( \Delta \) be the Laplacian with respect to the metric.

a) Show that for any smooth functions \( f, g \) on \( D \),
\[ \int_{\partial D} \left( f \frac{\partial g}{\partial r} - g \frac{\partial f}{\partial r} \right) = \int_D \left( f \Delta g - g \Delta f \right). \]

(here \( \partial/\partial r \) is the outward-pointing radial unit vector field, defined on \( D \setminus \{(0,0)\} \))

b) State and prove the generalization of a) to an arbitrary Riemannian manifold.
5. a) Give an example of a non-complete vector field on a manifold.
   b) Show that any left-invariant vector field on a Lie group is complete.

6. Let $(M, g)$ be a Riemannian manifold and let $\nabla$ denote the Riemannian connection. Given a one-form $\alpha \in \mathcal{A}^1(M)$ and a vector field $X \in T(M)$ define $\nabla_X \alpha$ by
   \[
   \nabla_X \alpha(Y) := X(\alpha(Y)) - \alpha(\nabla_X Y).
   \]
   a) Prove that $\nabla_X \alpha \in \mathcal{A}^1(M)$.
   b) Prove that $\nabla_X (f \alpha) = (X f) \alpha + f \nabla_X \alpha; \ f \in C^\infty(M)$.
   c) Suppose $f \in C^\infty(M)$ and $p \in M$ is a critical point of $f$. Compute $\nabla_X (df)(p)$ in local coordinates $(U, x_1, \ldots, x_n)$ around $p$.

7. Let $M = (0, \pi/2) \times \mathbb{R} \subset \mathbb{R}^2$ with the Riemannian metric
   \[
ds^2 = dx^2 + \cos^2(x)dy^2.
   \]
   a) Compute the Gaussian curvature of $(M, ds^2)$.
   b) Write the geodesic equations for $(M, ds^2)$. 