Let $C(X,Y)$ denote the set of continuous functions from topological spaces $X$ to $Y$. Let $\mathbb{R}$ denote the real line with the standard topology.

1. (a) Give an example of a space which is Hausdorff, locally-connected, and not connected.
   (b) Give an example of a space which is Hausdorff, path-connected, and not locally path-connected.

2. Consider the sequence $f_n(x) = \sin(\frac{x}{n})$. Determine whether or not the sequence converges in each of the following topologies of $C(\mathbb{R},\mathbb{R})$: the uniform topology, the compact-open topology, and the point-open (pointwise convergence) topology.

3. Let $X = \mathbb{R}/\mathbb{Z}$ be the quotient space where the integers $\mathbb{Z} \subset \mathbb{R}$ are identified to a single point. Prove that $X$ is connected, Hausdorff, and non-compact.

4. Prove the Uniform Limit Theorem: “Let $X$ be any topological space and $Y$ a metric space. Let $f_n \in C(X,Y)$ be a sequence which converges uniformly to $f$. Then $f \in C(X,Y)$.”

5. (a) Let $f : X \to Y$ be a quotient map, where $Y$ is connected. Suppose that for all $y \in Y$, $f^{-1}(y)$ is connected. Show that $X$ is connected.
   (b) Let $g : A \to B$ be continuous, where $B$ is path-connected. Suppose that for all $b \in B$, $g^{-1}(b)$ is path-connected. Suppose there exists $h \in C(B,A)$ such that $g \circ h$ is the identity on $B$. Show that $A$ is path-connected.

6. Let $\mathbb{R}_d$ and $\mathbb{R}_l$ denote the the real line with the the discrete topology and the lower limit topology, respectively. Recall that a basis for $\mathbb{R}_l$ is the set of intervals $[a,b)$ where $a < b$. List the functions that make up the following sets: $C(\mathbb{R},\mathbb{R}_d)$, $C(\mathbb{R}_d,\mathbb{R}_l)$, $C(\mathbb{R},\mathbb{R}_l)$.

7. Consider $\mathbb{R}^\omega$ with the product topology, the box topology and the uniform topology. Define the subset
   $$A = \{ (x_1, x_2, \ldots) \in \mathbb{R}^\omega \mid 0 < x_i < 1 \text{ for all } i \in \mathbb{N} \}.$$  
   In each topology, describe whether or not $A$ is open.