Do five of the following problems. All problems carry equal weight.

Passing Level:
Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1. Which of the following iterations will converge to the indicated fixed point \( \alpha \) (provided \( x_0 \) is sufficiently close to \( \alpha \))? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

   (a) \( x_{n+1} = -1 + x_n + \frac{2}{x_n}, \ \alpha = 2 \).
   (b) \( x_{n+1} = \frac{1}{2} x_n + \frac{3}{2x_n}, \ \alpha = 3^{1/3} \).
   (c) \( x_{n+1} = \frac{16}{1+x_n} - 1, \ \alpha = 3 \).

2. Determine values of \( \alpha \) in the matrix below

\[
\begin{pmatrix}
9 & 6 & 9 \\
6 & 20 & 26 \\
9 & 26 & \alpha \\
\end{pmatrix}
\]

for which the matrix is positive definite. (Hint: Use Cholesky Decomposition.)

3. Solve the following minimization problem and determine whether there is a unique \( \alpha \) that gives the minimum. \( \alpha \) is allowed to range over all reals.

\[
\min_{\alpha} \int_{-1}^{1} (x - \alpha x^2)^2 \, dx
\]

We are approximating \( f(x) = x \) by polynomials of the form \( \alpha x^2 \) in the continuous least squares sense.
4. Consider a well-posed system of ordinary differentiation equations

\[
\begin{align*}
\begin{cases}
  u_t &= Au \\
  u(0) &= u^0,
\end{cases}
\end{align*}
\]

where \( A \) is a constant matrix and \( u^0 \) is the initial condition.

(a) Write down the forward Euler scheme for such a system.
(b) Give a non-trivial condition for \( \Delta t \) which guarantees stability.
(c) For what class of matrices \( A \) is the condition \( \rho(A) \Delta t < 1 \), where \( \rho(A) \) is the spectral radius of \( A \), also a sufficient condition for the stability?

5. An \( n \times n \) matrix \( A \) is said to be strictly diagonally dominant if

\[
\sum_{j=1, j \neq i}^{n} |a_{ij}| < |a_{ii}| \quad \text{for } i = 1, \ldots, n.
\]

Note that the strict inequality implies that each diagonal entry \( a_{ii} \) is non-zero. Prove that the Jacobi iteration matrix \( B_J \) for \( A \) satisfies \( \|B_J\|_\infty < 1 \) and therefore the iteration converges in this case for any initial vector \( x^{(0)} \).

6. Consider the integral \( \int_{-1}^{1} f(x) \, dx \),

(a) Give the definition of Gauss quadrature
(b) Find the points and weights for the Gauss quadrature with exactly two points.

7. Consider the ODE initial-value problem

\[
\frac{dy}{dx}(x) = f(x, y(x)),
\]

with initial data \( y(x_0) = y_0 \). We would like to solve this initial value problem at points \( x_n = nh, n = 0, \ldots, N \) where \( h = x_n - x_{n-1} \) for all \( n \). Find the highest order method in the class

\[
y_{n+1} = y_n + h[b_1 f(x_n, y_n) + b_2 f(x_{n-1}, y_{n-1})].
\]

i.e., find \( b_1 \) and \( b_2 \) for the above method which gives the highest order local truncation error. State the order of the method obtained.