Department of Mathematics and Statistics
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Basic Exam - Complex Analysis

August 2007

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. (a) Find all values of $(i-1)^i$.

   (b) Let $f = u + iv$ be holomorphic in a connected open set A and where u, v are real-valued functions. Suppose there are real constants a, b, c such that $a^2 + b^2 \neq 0$ and $au + bv = c$ in A. Show that f is constant in A.

2. Suppose $b > 0$ is a positive real number. Use the Residue Theorem to compute the integral
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} dx.$$
Carefully justify all your steps, including the proofs of all the estimates.

3. (a) State and prove Rouche's Theorem.

   (b) Let f be holomorphic in an open set containing the closed unit disc. Suppose there is a real number m such that $|f(0)| < m < |f(z)|$ for all z with $|z| = 1$. Prove that $f(z)$ has at least one zero in the open unit disc $|z| < 1$.

4. Exhibit a one-to-one conformal map from $\{z \in \mathbb{C} \mid |z| < 1, \mathrm{Re}(z) > 0\}$ onto the unit disk $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

5. Let Ω be a connected open subset of the complex plane and $f_n(z)$, $n \geq 1$, a sequence of holomorphic and nowhere vanishing functions on Ω. Assume that the sequence$f_n(z)$ converges to a function $f(z)$, uniformly on every compact subset of Ω. Prove that f is either identically zero, or never equal to zero in Ω.

6. (a) Let C be the circle $\{z \ : \ |z - \pi| = \pi\}$ traversed once counterclockwise. Compute the integral $\int_C z \tan(z) dz$.

   (b) Let C be the circle $\{z \ : \ |z - 2| = 2\}$ traversed once counterclockwise. Compute the integral $\displaystyle\int_C \frac{z^2(5z-1)^8}{1 - z^{10}} dz$.

7. (a) Let C be the unit circle traversed once in the counterclockwise direction. Prove the equality
$$\left(\frac{t^n}{n!}\right)^2 = \frac{1}{2\pi i} \int_C \frac{t^n e^{tz}}{n! z^{n+1}} dz,$$
for any real number t. *Hint:* Multiply both sides by $\frac{(n!)^2}{t^n}$.

(b) Use part **??** to prove the identity:

$$\sum_{n=0}^{\infty}\left(\frac{t^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2t\cos(\theta)}d\theta.$$

8. Find a Laurent series that converges in $\{z \ : \ 1 < |z| < 2\}$ to a branch of the function $\log\left(\dfrac{z(2-z)}{(1-z)}\right)$.

9. Let f be a non-constant holomorphic function defined on a simply connected open subset U of the complex plane. Assume that all the zeroes of f are of even order. Prove that there exists a single valued holomorphic function g, satisfying $g^2(z) = f(z)$, for all $z \in U$ (so g is a square root of f). *Hint:* Consider an expression of the form $e^{\frac{1}{2}\int_{z_0}^{z}\frac{f'(\zeta)}{f(\zeta)}d\zeta}$. Carefully explain why your construction is holomorphic and single valued.

10. (a) State the Schwarz Lemma. (You are not asked to prove it).

    (b) Let $D = \{z \ : \ |z| < 1\}$ be the unit disk and $f : D \to D$ a one-to-one and onto holomorphic function. Prove that there exists a complex number $a \in D$ and a real number θ, such that $f(z) = e^{i\theta}\dfrac{a-z}{1-\bar{a}z}$.