DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. The growth of cancerous tumors can be modeled by Gompertz law $\dot{N} = -aN \ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.

   (a) Interpret a and b biologically.

   (b) Using linear stability analysis, classify the fixed points of the Gompertz model of tumor growth.

2. Here is a model for a love affair. Let $R(t)$ be Romeo's love/hate for Juliet at time t and let $J(t)$ be Juliet's love/hate for Romeo at time t. Positive values of R, J signify love, negative values signify hate. A model of this star-crossed romance is

$$\begin{aligned} \frac{dR}{dt} &= J \\ \frac{dJ}{dt} &= -R + J. \end{aligned}$$

   (a) Characterize the romantic styles of Romeo and Juliet.

   (b) Classify the fixed point at the origin. What does this imply for this love affair?

   (c) Sketch $R(t)$ and $J(t)$ as functions of t, assuming $R(0) = 1$ and $J(0) = 0$.

3. Let $x(t)$ be the number of rabbits at time t and $y(t)$ be the number of sheep at time t governed by the nonlinear system

$$\begin{aligned}\frac{dx}{dt} &= x(3-x-y) \\ \frac{dy}{dt} &= y(2-x-y).\end{aligned}$$

Find the fixed points, investigate stability and sketch a plausible phase portrait.

4. (a) Give a physical interpretation of the equation

$$u_t + xu_x = 0$$

(b) Draw the characteristics and solve the above equation with initial data $u(x,0) = cos(x)$.

5. Find all solutions to

$$u_{xx} + u_{yy} = 0$$

in the rectangle $0 < x < a$, $0 < y < b$, with the following boundary conditions

$$u_x = -a \text{ on } x = 0, \; u_x = 0 \text{ on } x = a.$$

$$u_y = b \text{ on } y = 0, \; u_y = 0 \text{ on } y = b.$$

(Hint: Try $u(x,y) = X(x) + Y(y)$ not $u(x,y) = X(x)Y(y)$.)

6. Consider a thin cylindrical metal bar with heat conductivity k. Suppose that the temperature at any point x along the bar is denoted by $\theta(x,t)$

at time t. x runs from $x = 0$ to $x = 1$, the length of the bar is one. The ends of the bar are maintained at temperatures θ_0 at $x = 0$ and θ_1 at $x = 1$

(a) Write down the governing partial differential equation and boundary conditions for this physical problem.

(b) Find the Fourier series solution when the initial temperature profile is

$$\theta(x, 0) = \theta_0 + (\theta_1 - \theta_0)x + \alpha x(1 - x)$$

for a constant $\alpha > 0$.