BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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Do 7 of the following 9 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. leve, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. Let A, B be real, $n \times n$ matrices such that $A^2 = A$ and $B^2 = B$. Suppose A and B have the same rank. Show that A and B are similiar.

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2. Denote by $M_{2\times 2}$ the real vector space of all 2×2 real matrices. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and denote by $\phi : M_{2\times 2} \to M_{2\times 2}$ the linear transformation defined by $\phi(M) = AM - MA$.

(a) Is ϕ diagonalizable?

(b) Is ϕ invertible?

Justify your answer!

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3. Let A be a real, $n \times n$ orthogonal matrix (i.e. $A^t A = I_n$, the $n \times n$ identity) and with $\det A = 1$.

(a) Show that every eigenvalue of A has absolute value 1.

(b) If n is odd, show that 1 is an eigenvalue of A.

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4. Let V, W be finite dimensional real vector spaces, and let $T : V \to W$ be a linear transformation. Determine

$$\dim(\ker T) + \dim(\mathrm{image} T).$$

Justify your answer!

Part II. Advanced Calculus

1. Let $f_1, f_2, \ldots$ be continuous functions on $[0,1]$ satisfying $f_1(x) \geq f_2(x) \geq \cdots$ and $\lim_{n\to\infty} f_n(x) = 1$ for all x. Prove or give a counterexample: the sequence of functions $\{f_n\}_n$ uniformly converges to the constant function 1 on $[0,1]$.

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2. Let $g : [1,\infty) \to \mathbf{R}$ be a function which is uniformly continuous. If $g(x) \geq 0$ for all x and if $\int_1^\infty g(t)dt$ exists and is finite, show that $\lim_{x\to\infty} g(x) = 0$.

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3. Evaluate

$$\alpha := \int_0^{1/2} \frac{\sin(t)}{t} dt$$

to two decimal places, i.e. find a real number β such that $|\alpha - \beta| < 0.005$. Show your work!

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4. Determine all values (a,b) for which the function

$$f_{a,b}(x,y) := ay^2 + bx$$

has exactly four critical points along the ellipse $3x^2 + 2y^2 = 1$.

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5. Denote by $\vec{F}$ the following vector field in $\mathbf{R}^3$

$$(x^2 + y - 4)\vec{\imath} + (3xy)\vec{\jmath} + (2xz + z^2)\vec{k}.$$

(a) Compute $\nabla \times \vec{F}$ (in other words, curl$\vec{F}$).
(b) Compute the integral of $\nabla \times \vec{F}$ along the surface $x^2 + y^2 + z^2 = 25$ with $z \geq 3$, oriented so that the normal vectors point *towards the origin.*

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6. Denote a sequence $\{a_n\}$ recursively as follow:

$$a_1 = 3, \;\; a_{n+1} = \sqrt{3 + a_n} \;\; (n \geq 1)$$

Show that this sequences converges to a finite number and determine this number. Show your work!
