Do 7 of the following 9 problems.

**Passing Standard:** For Master’s level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

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**Part I. Linear Algebra**

1. Let $A, B$ be real, $n \times n$ matrices such that $A^2 = A$ and $B^2 = B$. Suppose $A$ and $B$ have the same rank. Show that $A$ and $B$ are similar.

2. Denote by $M_{2 \times 2}$ the real vector space of all $2 \times 2$ real matrices. Let

   
   $$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   and denote by $\phi : M_{2 \times 2} \rightarrow M_{2 \times 2}$ the linear transformation defined by $\phi(M) = AM - MA$.

   (a) Is $\phi$ diagonalizable?
   (b) Is $\phi$ invertible?

   Justify your answer!

3. Let $A$ be a real, $n \times n$ orthogonal matrix (i.e. $A^tA = I_n$, the $n \times n$ identity) and with $\det A = 1$.

   (a) Show that every eigenvalue of $A$ has absolute value 1.
   (b) If $n$ is odd, show that 1 is an eigenvalue of $A$.

4. Let $V, W$ be finite dimensional real vector spaces, and let $T : V \rightarrow W$ be a linear transformation. Determine

   $$\dim(\ker T) + \dim(\text{image}T).$$

   Justify your answer!
Part II. Advanced Calculus

1. Let $f_1, f_2, \ldots$ be continuous functions on $[0, 1]$ satisfying $f_1(x) \geq f_2(x) \geq \cdots$ and $\lim_{n \to \infty} f_n(x) = 1$ for all $x$. Prove or give a counterexample: the sequence of functions $\{f_n\}_n$ uniformly converges to the constant function 1 on $[0, 1]$.

2. Let $g : [1, \infty) \to \mathbb{R}$ be a function which is uniformly continuous. If $g(x) \geq 0$ for all $x$ and if $\int_1^\infty g(t) dt$ exists and is finite, show that $\lim_{x \to \infty} g(x) = 0$.

3. Evaluate
$$\alpha := \int_0^{1/2} \frac{\sin(t)}{t} dt$$
to two decimal places, i.e. find a real number $\beta$ such that $|\alpha - \beta| < 0.005$. Show your work!

4. Determine all values $(a, b)$ for which the function
$$f_{a,b}(x, y) := ay^2 + bx$$
has exactly four critical points along the ellipse $3x^2 + 2y^2 = 1$.

5. Denote by $\vec{F}$ the following vector field in $\mathbb{R}^3$
$$(x^2 + y - 4)\vec{i} + (3xy)\vec{j} + (2xz + z^2)\vec{k}.$$ (a) Compute $\nabla \times \vec{F}$ (in other words, curl$\vec{F}$).
(b) Compute the integral of $\nabla \times \vec{F}$ along the surface $x^2 + y^2 + z^2 = 25$ with $z \geq 3$, oriented so that the normal vectors point towards the origin.

6. Denote a sequence $\{a_n\}$ recursively as follow:
$$a_1 = 3, \quad a_{n+1} = \sqrt{3 + a_n} \quad (n \geq 1)$$
Show that this sequences converges to a finite number and determine this number. Show your work!