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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
August, 2007

Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1. Describe the global behavior of all solutions of the system

$$\begin{aligned} \frac{dx}{dt} &= -y - x(x^2+y^2) \\ \frac{dy}{dt} &= x - y(x^2+y^2). \end{aligned}$$

Does the linearized system of the rest point at the origin correctly predict the behavior of small solutions of the nonlinear equations? Justify your answer with an appropriate calculation and explanation.

2. Let u be a C^2 function defined on the set $\Omega = [0,1]^n \subset \mathbb{R}^n$, and satisfying the conditions

$$|\Delta u| \leq 1, \quad \text{and} \quad u\big|_{\partial\Omega} = 0.$$

Find upper and lower bounds for u on Ω.

3. Let $(\theta(t), \psi(t))$ be the solution of the system

$$\begin{aligned} \frac{d\theta}{dt} &= \psi \\ \frac{d\psi}{dt} &= -c\,\psi - \sin\theta \end{aligned}$$

where c is a nonnegative constant.

(a) Sketch the global phase plane for the differential equations when $c = 0$; your sketch must be accompanied by appropriate analytical calculations needed to verify qualitative behaviors such as the presence of periodic, heteroclinic, or homoclinic orbits and the behavior of solutions near rest points.

(b) Answer the same question as in part (a) when $c = 1$.

(c) Suppose $(\phi(t), \psi(t))$ is the solution of the above system when $c = 1$ and with the initial condition $\theta(0) = -\pi$ and $\psi(0) = p$ where p is a positive number. Prove that there exists a nonnegative integer $N \geq 0$ (depending on p) such that

$$\lim_{t\to+\infty} (\theta(t), \psi(t)) = (N\pi, 0).$$

4. Consider the form $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$B(u,v) = \int_\Omega Du\; Dv\; dx + \int_\Omega u\; v\; dx + \int_{\partial\Omega} \alpha(s)\; \mathcal{T}u\; \mathcal{T}v\; ds,$$

where $\alpha \in L^\infty(\partial\Omega)$ and $\mathcal{T} : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace map. Under what condition on α will B be coercive on H^1 ? In this case, prove there exists a unique weak solution to

$$-\Delta u + u = f \quad \text{in } \Omega,$$

with boundary condition

$$\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \partial\Omega,$$

for every $f \in L^2(\Omega)$.

5. Given smooth functions $f(x,y)$, $g(x,y)$ in a rectangle $R = [A,B] \times [C,D]$, let $(F(x,y), G(x,y))$ be the functions defined by

$$\begin{aligned} F(x,y) &= Max\{f(x,y_1),\;\; C \le y_1 \le y\} \\ G(x,y) &= Max\{g(x_1,y),\;\; A \le x_1 \le x\}. \end{aligned}$$

(a) Show that the initial value problem

$$\begin{aligned} \frac{dX}{dt} &= F(X,Y), \;\; X(0) = X_* \\ \frac{dY}{dt} &= G(X,Y), \;\; Y(0) = Y_* \end{aligned}$$

(with $(X_*, Y_*) \in R$) has a unique local solution $(X(t), Y(t))$ for small t; (you need only cite without proof the main existence and uniqueness theorem for ODE's, but you need to verify its hypotheses for this particular system.)

(b) Let $(x(t), y(t))$ be a solution of the ODE's

$$\begin{aligned} \frac{dx}{dt} &= f(x,y), \;\; x(0) = x_o \\ \frac{dy}{dt} &= g(x,y), \;\; y(0) = y_o. \end{aligned}$$

Show that if $(x(t), y(t))$ and $(X(t), Y(t))$ both exist on an interval $0 \le t \le T$ and that $x_o < X_*$, $y_o < Y_*$ then $\xi(t) = X(t) - x(t)$ and $\eta(t) = Y(t) - y(t)$ are both nonnegative functions for $0 \le t \le T$. (*Hint*: show that the quadrant $\xi, \eta \ge 0$ is an invariant region for the differential equations satisfied by $(\xi(t), \eta(t))$.

6. Consider the system

$$\begin{aligned} u_t + x^2\, v_x &= 0, \quad (1) \\ v_t + x^2\, u_x &= 0, \quad (2) \end{aligned}$$

with initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x).$$

(a) Use the method of characteristics to find the general solution to the system.

(b) On what region is the solution defined ?