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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M})$ be a measure space and $\{f_n, n \in \mathbb{N}\}$ a sequence of measurable functions mapping X into $\mathbb{R}$.

   (a) For each $x \in X$, define

   $$g(x) = \limsup_{n\to\infty} f_n(x) \text{ and } h(x) = \liminf_{n\to\infty} f_n(x).$$

   Assume that for all $x \in X$, $g(x)$ and $h(x)$ are finite. Prove that g and h are both measurable functions on $(X, \mathcal{M})$.

   (b) Assume that for all $x \in X$

   $$\varphi(x) = \lim_{n\to\infty} f_n(x)$$

   exists and is finite. Prove that φ is a measurable function on $(X, \mathcal{M})$.

2. For A a subset of the set $\mathbb{N}$ of positive integers, define $\mu(A)$ to be the cardinality of A.

   (a) Prove that μ is a measure on the σ-algebra of all subsets of $\mathbb{N}$.

   (b) Prove that μ is a σ-finite measure.

   (c) Let f be any nonnegative function mapping $\mathbb{N}$ into $[0, \infty)$. Prove that $\int_{\mathbb{N}} f\, d\mu = \sum_{j=1}^{\infty} f(j)$.

   (d) For $j \in \mathbb{N}$ and $n \in \mathbb{N}$ define
   $$f_n(j) = \frac{1}{2^j}\left(1 - \frac{1}{n}\right).$$
   Using a well known limit theorem and part (c), evaluate $\lim_{n\to\infty} \sum_{j=1}^{\infty} f_n(j)$.

   (e) For $j \in \mathbb{N}$ and $n \in \mathbb{N}$ define
   $$g_n(j) = \frac{1}{2^j}\left(1 + \frac{1}{n}\right).$$
   Using a well known limit theorem and part (c), evaluate $\lim_{n\to\infty} \sum_{j=1}^{\infty} g_n(j)$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space; f and g positive measurable functions mapping X into $(0, \infty)$; and t, r, and m real numbers satisfying $0 < t < r < m < \infty$.

   (a) Roger's inequality states that if both integrals on the right hand side of the following display are finite, then

$$\left(\int_X fg^r \, d\mu\right)^{m-t} \leq \left(\int_X fg^t \, d\mu\right)^{m-r} \left(\int_X fg^m \, d\mu\right)^{r-t}.$$

   Prove Roger's inequality using Hölder's inequality. **Hint:** Identify the conjugate exponents p and q and use the identity

$$r = \frac{m-r}{m-t}\, t + \frac{r-t}{m-t}\, m.$$

   (b) Let p and q be real numbers satisfying $1 < p, q < \infty$ and $1/p + 1/q = 1$. Let $\varphi \in L^p(\mu)$ and $\psi \in L^q(\mu)$ be positive functions mapping X into $(0, \infty)$. Use Roger's inequality to prove that

$$\|\varphi\psi\|_{L^1(\mu)} \leq \|\varphi\|_{L^p(\mu)} \, \|\psi\|_{L^q(\mu)}.$$

   (**Hint.** In Roger's inequality let $t = 1$ and $m = 2$ and choose f and g appropriately.)

4. Let $(X, \mathcal{M})$ be a measurable space.

(a) Let ν be a measure on $(X, \mathcal{M})$. Assume that there exists a nonnegative measurable function g on $(X, \mathcal{M})$ having the property that for all $A \in \mathcal{M}$, $\nu(A) = \int_A g \, d\nu$. Prove that $g = 1$ a.e.

(b) Let ρ and λ be σ-finite measures on $(X, \mathcal{M})$ having the property that $\rho \ll \lambda$ and $\lambda \ll \rho$. Prove that the Radon-Nikodym derivatives satisfy

$$d\rho/d\lambda = \frac{1}{d\lambda/d\rho} \text{ a.e. with respect to either } \rho \text{ or } \lambda.$$

5. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $(\mathbb{R}, \mathcal{L}, m)$ the real line equipped with the Lebesgue σ-algebra and Lebesgue measure. The product space is denoted by $(X \times \mathbb{R}, \mathcal{M} \otimes \mathcal{L}, \mu \times m)$.

   (a) Let f be a measurable function mapping X into $\mathbb{R}$ and define the function $\hat{f}$ mapping $X \times \mathbb{R}$ into $\mathbb{R}$ by $\hat{f}(x, y) = f(x)$ for $(x, y) \in X \times \mathbb{R}$. Prove that $\hat{f}$ is measurable with respect to $\mathcal{M} \otimes \mathcal{L}$.

   (b) Let g be a nonnegative function mapping X into $[0, \infty)$. Define the set

$$A_g = \{(x, y) \in X \times \mathbb{R} : 0 \leq y \leq g(x)\}.$$

   Prove that g is a measurable function if and only if A_g is a measurable subset of $X \times \mathbb{R}$.

   (c) Let g be a nonnegative measurable function mapping X into $[0, \infty)$ and let A_g be the subset of $X \times \mathbb{R}$ defined in part (b). Prove that

$$\int_X g \, d\mu = \mu \times m(A_g).$$

6. (a) Let φ and ψ be absolutely continuous functions on a finite closed interval $[a, b]$. Prove that the product $\varphi\psi$ is absolutely continuous.

(b) Let f and g be integrable functions on a finite closed interval $[a, b]$. For $x \in [a, b]$ define

$$F(x) = \alpha + \int_a^x f(t)\, dt \text{ and } G(x) = \beta + \int_a^x g(t)\, dt,$$

where α and β are fixed real numbers. Using part (a), prove that

$$\int_a^b G(t)f(t)dt + \int_a^b F(t)g(t)dt = F(b)G(b) - F(a)G(a).$$

7. Let $\mathcal{H}$ be a real vector space and let $\langle \cdot, \cdot \rangle$ be a real inner product on $\mathcal{H}$ with norm $\|x\| = \langle x, x \rangle^{1/2}$.

(a) For all x and y in $\mathcal{H}$ prove the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

and the polarization identity

$$\langle x, y \rangle = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right).$$

(b) Let $\| \cdot \|$ be a norm on $\mathcal{H}$ satisfying the parallelogram law. For x and y in $\mathcal{H}$ define $\langle x, y \rangle$ by the polarization identity stated in part (a).

(i) Prove that for all x, y, and z in $\mathcal{H}$, $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(ii) Use part (b)(i) to prove that for all $\alpha \in \mathbb{R}$ and all x and y in $\mathcal{H}$, $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$. (**Hint.** First prove this for integers α, then for rational numbers α, and finally for real numbers α.)

8. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $\{D_i, i = 1, 2, \ldots, k\}$ be a finite, disjoint collection of sets in $\mathcal{M}$ satisfying $\mu(D_i) > 0$ for all $i = 1, 2, \ldots, k$. Define $\mathcal{D}$ to be the σ-algebra generated by $\{D_i, i = 1, 2, \ldots, k\}$ and define $\mathcal{W}_\mathcal{D}$ to be the set of $\mathcal{D}$-measurable simple functions that map X into $\mathbb{R}$ (all finite linear combinations, with real coefficients, of indicator functions of sets in $\mathcal{D}$). $\mathcal{W}_\mathcal{D}$ is a closed subspace of the real Hilbert space $L^2(X, \mathcal{M}, \mu)$ consisting of all square integrable functions mapping X into $\mathbb{R}$ and equipped with the usual inner product and norm.

(a) Prove that the functions $\{1_{D_i}, i = 1, 2, \ldots, k\}$ form an orthogonal basis of $\mathcal{W}_\mathcal{D}$.

(b) Find an orthonormal basis of $\mathcal{W}_\mathcal{D}$.

(c) For $Y \in L^2(\mathcal{M})$ calculate, in terms of your answer to part (b), the orthogonal projection of Y onto $\mathcal{W}_\mathcal{D}$.

(d) Let $(X, \mathcal{M}, \mu) = ([0, 1], \mathcal{B}[0, 1], m)$ and $k = 3$. Define the intervals $D_1 = [0, 1/3)$, $D_2 = [1/3, 2/3)$, and $D_3 = [2/3, 1]$. For $x \in [0, 1]$ define $f(x) = x$. Calculate explicitly the function $g_0 \in \mathcal{W}_\mathcal{D}$ satisfying

$$\|f - g_0\| = \inf\{\|f - g\| : g \in \mathcal{W}_\mathcal{D}\}.$$

Indicate what theorem(s) about Hilbert space you are using in your answer.