Passing Standard: It is sufficient to do five problems correctly, including at least one from each of the four parts.

I: Group theory
1. Let $G$ be a finite group. Assume that there is $g \in G$ with conjugacy class consisting of exactly two elements. Show that $G$ contains a non-trivial proper normal subgroup $N$.
2. Prove that (up to isomorphism) there is a unique non-abelian group of order $2007 = 3^2 \cdot 223$ containing an element of order 9.

II: Ring theory
3. Let $R$ denote the field of real numbers. Let $A$ denote a commutative $R$-algebra which is two-dimensional as an $R$-vector space. (Recall that this simply means that $A$ is a commutative ring containing $R$ as a subring; $A$ then becomes an $R$-vector space in the obvious way, and we are assuming that it has dimension two.) Prove that $A$ is isomorphic to one of the three rings: $R \times R$, $\mathbb{C}$, $R[x]/(x^2)$.
4. Let $R$ be a commutative ring. Let $I, J_1, J_2$ be ideals of $R$.
   (a) Show that if $I \subseteq J_1 \cup J_2$, then $I \subseteq J_1$ or $I \subseteq J_2$.
   (b) Let $P$ be a prime ideal of $R$. Show that if $I \subseteq J_1 \cup J_2 \cup P$, then $I \subseteq J_1$ or $I \subseteq J_2$ or $I \subseteq P$.

III: Modules
5. Let $R$ be a principal ideal domain and let $A, B, C$ be torsion (i.e., rank 0) $R$-modules. Prove that if $\text{Hom}_R(A \otimes_R B, C) \neq 0$, then there is a non-zero prime ideal $P$ of $R$ such that each of the modules $A/P A, B/P B, C/P C$ is non-zero.
6. Determine all similarity classes of $3 \times 3$ matrices $A$ over $\mathbb{F}_2$ satisfying $A^6 = I$.

IV: Field theory
7. Fix a prime $p$ and let $F_{p^2}$ denote the field with $p^2$ elements.
   (a) Define an injective ring homomorphism
   $$\varphi : F_{p^2} \hookrightarrow M_2(F_p)$$
   with $M_2(F_p)$ the ring of $2 \times 2$ matrices over $F_p$. (Hint: choose a basis for $F_{p^2}$ over $F_p$.)
   (b) For which $\alpha \in F_{p^2}$ is $\varphi(\alpha)$ diagonalizable over $F_p$?
   (c) Is there $\alpha \in F_{p^2}$ such that $\varphi(\alpha)$ is similar (over $\overline{F}_p$) to a matrix
   $$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$
   with $\lambda \in \overline{F}_p$?
8. Let $L/K$ be a Galois extension of fields with Galois group isomorphic to the symmetric group $S_4$. For which integers $n$ do there exist $\alpha \in L$ of degree $n$ over $K$? Justify your answer.