Advanced Exam – Algebra
August 2007

Passing Standard: It is sufficient to do **five** problems correctly, including at least **one from each** of the **four** parts.

I: Group theory

1. Let G be a finite group. Assume that there is $g \in G$ with conjugacy class consisting of exactly two elements. Show that G contains a non-trivial proper normal subgroup N.

2. Prove that (up to isomorphism) there is a unique non-abelian group of order $2007 = 3^2 \cdot 223$ containing an element of order 9.

II: Ring theory

3. Let $\mathbf{R}$ denote the field of real numbers. Let A denote a commutative $\mathbf{R}$-algebra which is two-dimensional as an $\mathbf{R}$-vector space. (Recall that this simply means that A is a commutative ring containing $\mathbf{R}$ as a subring; A then becomes an $\mathbf{R}$-vector space in the obvious way, and we are assuming that it has dimension two.) Prove that A is isomorphic to one of the three rings: $\mathbf{R} \times \mathbf{R}$, $\mathbf{C}$, $\mathbf{R}[x]/(x^2)$.

4. Let R be a commutative ring. Let I, J_1, J_2 be ideals of R.
   (a) Show that if $I \subseteq J_1 \cup J_2$, then $I \subseteq J_1$ or $I \subseteq J_2$.
   (b) Let P be a prime ideal of R. Show that if $I \subseteq J_1 \cup J_2 \cup P$, then $I \subseteq J_1$ or $I \subseteq J_2$ or $I \subseteq P$.

III: Modules

5. Let R be a principal ideal domain and let A, B, C be torsion (i.e., rank 0) R-modules. Prove that if

$$\mathrm{Hom}_R(A \otimes_R B, C) \neq 0,$$

   then there is a non-zero prime ideal P of R such that each of the modules $A/PA, B/PB, C/PC$ is non-zero.

6. Determine all similarity classes of 3×3 matrices A over $\mathbf{F}_2$ satisfying $A^6 = I$.

IV: Field theory

7. Fix a prime p and let $\mathbf{F}_{p^2}$ denote the field with p^2 elements.
   (a) Define an injective ring homomorphism

$$\varphi : \mathbf{F}_{p^2} \hookrightarrow M_2(\mathbf{F}_p)$$

   with $M_2(\mathbf{F}_p)$ the ring of 2×2 matrices over $\mathbf{F}_p$. (Hint: choose a basis for $\mathbf{F}_{p^2}$ over $\mathbf{F}_p$.)
   (b) For which $\alpha \in \mathbf{F}_{p^2}$ is $\varphi(\alpha)$ diagonalizable over $\mathbf{F}_p$?
   (c) Is there $\alpha \in \mathbf{F}_{p^2}$ such that $\varphi(\alpha)$ is similar (over $\overline{\mathbf{F}}_p$) to a matrix

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

   with $\lambda \in \overline{\mathbf{F}}_p$?

8. Let L/K be a Galois extension of fields with Galois group isomorphic to the symmetric group S_4. For which integers n do there exist $\alpha \in L$ of degree n over K? Justify your answer.