1. Consider the circuit equation

\[ LI'' + RI' + IC = 0 \]

where \( L, C > 0 \) and \( R \geq 0 \).

(a) Rewrite the equation as a two-dimensional system.

(b) Show that the origin is asymptotically stable if \( R > 0 \) and neutrally stable if \( R = 0 \).

(c) Classify the fixed point at the origin, depending on whether \( R^2C - 4L \) is positive, negative, or zero, and sketch the phase portrait in all three cases.

2. Consider the system \( x' = y^3 - 4x, \ y' = y^3 - y - 3x \).

(a) Find all the fixed points and classify them.

(b) Show that \( |x(t) - y(t)| \) approaches 0 as \( t \) approaches \( \infty \) for all other trajectories. (Hint: Form a differential equation for \( x - y \).)

(c) Draw the phase portrait.

3. In a certain fishery, assume that fish are caught at a constant rate \( h \) (harvesting rate) independent of the size of the fish population. \( K \) is
the natural capacity of the fishery, \( r \) is the natural growth rate. Then the number of fish in the fishery at any time \( t \), \( y(t) \), satisfies

\[
\frac{dy}{dt} = r(1 - \frac{y}{K})y - h
\]

(a) Determine a condition (an inequality between \( h, r, K \)) such that any initial fish population will eventually become depleted (that is, \( y(t) = 0 \) for some \( t > 0 \)).

(b) On the other hand, under what conditions is there a stable fixed point \( y^* \)? Give an explicit formula for \( y^* \).

4. Consider the initial boundary value problem for a function \( u(x, t) \):

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0
\]

\[
u(L, t) = B, \quad \frac{\partial u}{\partial x}(0, t) = Q
\]

\[
u(x, 0) = u_0(x).
\]

(a) Explain in physical terms the meaning of the constants \( D \), \( B \), \( Q \) when \( u \) represents the temperature in a rod over the interval \( 0 \leq x \leq L \).

(b) Determine the equilibrium solution \( u^*(x) \) that is independent of time.

(c) The general solution with initial solution \( u_0(x) \) has the form

\[
u(x, t) = u^*(x) + \sum_{k=1}^{\infty} e^{-\lambda_k t} \phi_k(x).
\]

Exhibit both the differential equation and the boundary conditions that each function \( \phi_k \) must satisfy.
5. Consider the Laplace equation
\[ \Delta u = u_{xx} + u_{yy} = 0 \text{ in } x^2 + y^2 < R^2 \]
in a disk of radius \( R \). Find the solution \( u \) satisfying the boundary condition
\[ u(R, \theta) = 3 \cos(2\theta) + 5 \sin(\theta) \quad ( \theta \in [0, 2\pi] ) \]

6. The motion of a string with friction is modeled by the modified wave equation
\[ u_{tt} - c^2 u_{xx} + \gamma u_t = 0. \]
Here \( \gamma > 0 \) and \( u_x(0, t) = u_x(L, t) = 0 \).
(a) Let
\[ E = \frac{1}{2} \int_0^L (u_t^2 + c^2 u_x^2) \, dx \]
and derive the identity
\[ \frac{\partial E}{\partial t} = -\gamma \int_0^L u_t^2 \, dx \]
(b) Interpret this identity in terms of dissipation of energy.

7. Consider the following hat function \( f(x) \) given by
\[ f(x) = \begin{cases} x & \text{if } 0 \leq x \leq \pi/2 \\ \pi - x & \text{if } \pi/2 \leq x \leq \pi \end{cases} \]
(a) Find the Fourier sine series for \( f(x) \).
(b) Find the Fourier sine series for \( f'(x) \).
(c) What can you say about their convergence at \( \pi/2 \).