Advanced Calculus/Linear Algebra Basic Exam
August 2006

Do 7 of the following 9 problems. Indicate clearly on your answer booklet which problems should be graded.

Passing standard: For Master's level, 60% with three questions essentially correct (including at least one from each part). For Ph.D. level, 75% with two questions from each part essentially complete.

Part I: Linear algebra

1. Determine the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

2. Let V be a five-dimensional real vector space and let $T : V \to V$ be a linear transformation with characteristic polynomial $(x-3)^2(x+2)^3$.

   (a) Compute the determinant of T.

   (b) Do there exist linearly independent vectors $v_1, v_2 \in V$ such that $T(v_1), T(v_2)$ are also linearly independent?

   (c) Show that V possesses T-invariant subspaces of each dimension 1, 2, 3 and 4.

3. Let T and U be commuting linear transformations (that is, $TU = UT$) on a finite dimensional complex vector space V. Show that T and U have a simultaneous eigenvector: that is, show that there is a non-zero $v \in V$ which is an eigenvector for each of T and U.

4. Let $T : V \to V$ be a linear transformation of a finite-dimensional real vector space V. Suppose that T has no real eigenvalues. Show that every T-invariant subspace W of V has even dimension.

Part II: Advanced calculus

1. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{F}(x,y,z) = \langle x^3, x^2y, x^2z\rangle$ and S is the (closed) surface of the cylinder bounded by $x^2+y^2=1$, $z=0$ and $z=1$, oriented outwards.

2. Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a convergent series and let $\{b_n\}_{n=1}^{\infty}$ be a bounded, positive, increasing sequence. Prove that $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ converges.

3. Let f be a function whose second derivative $f''(x)$ exists and is continuous on an open interval (a,b). Prove that

$$f''(x) = \lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

for all $x \in (a,b)$.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a convex (and thus continuous) function: that is,

$$f\big(t_1x_1 + \cdots + t_nx_n\big) \le t_1 f(x_1) + \cdots + t_n f(x_n)$$

for all $x_1,\ldots,x_n \in \mathbf{R}$ and all $t_1,\ldots,t_n \ge 0$ such that $t_1+\cdots+t_n = 1$. Let $g(x)$ be a continuous function on $[0,1]$. Prove that

$$f\left(\int_0^1 g(x)\,dx\right) \le \int_0^1 f\big(g(x)\big)\,dx.$$

5. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the vector field

$$\mathbf{F}(x,y,z) = \left\langle y\tan^{-1} z,\ x\tan^{-1} z,\ \frac{xy}{1+z^2}\right\rangle$$

and the line C connecting $(0,0,0)$ to $(1,1,1)$.