University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1. Suppose M^2 is a connected, orientable, smooth surface and α is a l-form on M. Can $\omega := d\alpha$ be a non-vanishing 2-form when M is:

(1) compact without boundary?
(2) noncompact?

Give a proof or construct a (counter) example.

Problem 2. Let S^n denote the n-sphere, and let $f : S^n \to \mathbb{R}^k$ be a smooth map with regular value 0. Show that $X := f^{-1}(0)$ is an embedded submanifold of codimension k in S^n, with trivial normal bundle ν_X. Conversely, show that any such $X \subset S^n$ with ν_X trivial arises this way. (Hint: construct such an $f : S^n \to \mathbb{R}^k$ under these hypotheses.)

Problem 3. Let $S \subset \mathbb{R}^3$ be a smooth surface with Gauss curvature $K \equiv 0$. Show that S is locally foliated by straight lines.

Problem 4. Show that every smooth manifold admits a Riemannian metric.

Problem 5. Let

$$\mathbf{Sp}(n) := \{A \in \mathbf{Gl}(2n, \mathbb{R})\,;\, A^t J A = J\}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I & 0 \end{pmatrix}$. Show that $\mathbf{Sp}(n)$ is a Lie group, compute its Lie algebra and its dimension.

Problem 6. Let $\alpha \in \Omega^1(M, \mathbb{R})$ be a nowhere vanishing 1-form on a smooth manifold M such that $d\alpha \wedge \alpha = 0$. Show that each point $p \in M$ has a neighborhood $U \subset M$ and smooth maps $f, g : U \to \mathbb{R}$ such that $g\alpha = df$.

Problem 7. Let M be a manifold with conformally related Riemannian metrics g and $\tilde{g} = e^u g$.

(1) Find a formula expressing the Levi-Civita connections ∇ and $\tilde{\nabla}$ of g and $\tilde{g}$. Recall that the Levi-Civita connection is the unique torsion free connection on TM satisfying $\nabla g = 0$.
(2) If $N \subset M$ is a submanifold then $TM_{|N} = TN \oplus TN^{\perp}$ where $TN^{\perp}$ denotes the normal bundle of $N \subset M$ (note that $\perp$ is the same for conformally related metrics). Denote by $\nabla^{\perp}$ the connection on the normal bundle induced by the splitting of $TM_{|N}$. Show that
$$\tilde{\nabla}^{\perp}\xi = \nabla^{\perp}\xi + \xi\, du$$
for sections $\xi \in \Gamma(TM^{\perp})$.
(3) Conclude that the curvature of the normal bundle is a conformal invariant under conformal changes of metrics on M.

Problem 8. Let $M = \mathbb{R}^2$ and $\omega \in \Omega^1(M, \mathbf{gl}(n, \mathbb{R}))$ a matrix valued 1-form on M satisfying
$$d\omega + \omega \wedge \omega = 0\,.$$
Show that there exists a map $F : M \to \mathbf{Gl}(n, \mathbb{R})$ satisfying $dF = F\omega$. Moreover, any two such maps F_i satisfy $F_2 = gF_1$ with $g \in \mathbf{Gl}(n, \mathbb{R})$. Use this to show that any closed 1-form $\alpha \in \Omega^1(M, \mathbb{R})$ is of the form $\alpha = df$ for some function $f : M \to \mathbb{R}$ (special case of the Poincare Lemma).