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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Let $\{A_n, \in \mathbb{N}\}$ be a nondecreasing sequence in $\mathcal{M}$; i.e., $A_n \subset A_{n+1}$ for all n. Prove that

$$\lim_{n\to\infty} \mu(A_n) = \mu(\cup_{n\in\mathbb{N}} A_n).$$

(b) Let $\{A_n, \in \mathbb{N}\}$ be an arbitrary sequence in $\mathcal{M}$. Give the definition of the set $\liminf_{n\to\infty} A_n$ and prove that

$$\mu\Big(\liminf_{n\to\infty} A_n\Big) \leq \liminf_{n\to\infty} \mu(A_n).$$

2. (a) State Fatou's Lemma.

   (b) State the Dominated Convergence Theorem.

   (c) Prove the Dominated Convergence Theorem from Fatou's Lemma. (**Hint.** Consider $g + f_n$ and $g - f_n$).

3. (a) Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n, n \in \mathbb{N}\}$ a sequence of Borel-measurable functions mapping X into $\mathbb{R}$, and f a Borel-measurable function mapping X into $\mathbb{R}$. Assume that $f_n \to f$ in measure and that there exists $g \in L^1(\mu)$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Prove that $f_n \to f$ in $L^1(\mu)$; i.e., prove that

$$\lim_{n\to\infty} \int_X |f - f_n| \, d\mu = 0.$$

(**Hint.** Work with an arbitrary subsequence of $\{f_n\}$ that converges to f in measure. Alternatively, consider a proof by contradiction.)

(b) Give an example of a measure space $(X, \mathcal{M}, \mu)$, a sequence $\{f_n, n \in \mathbb{N}\}$ of Borel-measurable functions mapping X into $\mathbb{R}$, and a Borel-measurable function f mapping X into $\mathbb{R}$ with the following property: $f_n \to f$ in measure but f_n does not converge to f in $L^1(\mu)$.

4. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces.

(a) State the definition of the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$.

(b) Let μ be a finite measure on $(X, \mathcal{M})$ and let ν be a finite measure on $(Y, \mathcal{N})$. For $E \in \mathcal{M} \otimes \mathcal{N}$ and $x \in \mathcal{M}$, state the definition of the x-section E_x. Also state the formula expressing $\mu \times \nu(E)$ as an integral involving μ, ν, and E_x. Only state this formula; do not prove it.

(c) Let μ_1 and μ_2 be finite measures on $(X, \mathcal{M})$ and let ν_1 and ν_2 be finite measures on $(Y, \mathcal{N})$. Assume that $\mu_1 \ll \mu_2$ and $\nu_1 \ll \nu_2$. Prove that $\mu_1 \times \nu_1 \ll \mu_2 \times \nu_2$. (**Hint.** Use the formula in part (b).)

5. Given $-\infty < a < b < \infty$, let I be the closed, bounded interval $[a, b]$. Let φ be a **convex** function mapping I into $\mathbb{R}$. Fixing $x_0 \in I$, define

$$h(s) = \frac{\varphi(s) - \varphi(x_0)}{s - x_0}.$$

Prove that $h(s) \leq h(t)$ for all $s \in I$ and $t \in I$ satisfying $s < t$, $s \neq x_0$, and $t \neq x_0$.

6. Let H be a real, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, $\{e_k, k \in \mathbb{N}\}$ a countable orthonormal basis for H, x an element of H, and $\{x_n, n \in \mathbb{N}\}$ a bounded sequence in H. Thus there exists $M \in (0, \infty)$ such that $\|x_n\| \leq M$ for all n. Also let H^* denote the set of bounded linear functionals Φ mapping H into $\mathbb{R}$. For any $\Phi \in H^*$, prove that

$$\lim_{n \to \infty} \Phi(x_n) = \Phi(x) \text{ if and only if for all } k \in \mathbb{N}, \lim_{n \to \infty} \langle x_n, e_k \rangle = \langle x, e_k \rangle.$$

(**Hints.** In order to prove one direction of the implication, use the Riesz representation theorem, which states that for any $\Phi \in H^*$, there exists $y_\Phi \in H$ such that $\Phi(x) = \langle x, y_\Phi \rangle$ for all $x \in H$. Then approximate y_Φ by an appropriate partial sum and work with this partial sum.)

7. Let X be a Banach space with norm $\|\cdot\|$ and let E be a proper, nonempty, **closed** subspace of X. We define the following equivalence relation on X: $x \sim y$ iff $x - y \in E$. The equivalence class of $x \in X$ is denoted by $x + E$, and the set of equivalence classes, or quotient space, is denoted by X/E. With these definitions, X/E is a vector space (do not prove this). For $x \in X$, define

$$\|x + E\| = \inf_{y \in E} \|x + y\|.$$

(a) Prove that $\|x + E\|$ defines a norm on X/E.

(b) Prove that X/E is complete with respect to the norm $\|x + E\|$. (**Hint.** Use without proof the fact that a normed vector space Y is complete if and only if every absolutely convergent series in Y converges to an element in Y.)

8. Given $1 \leq p < \infty$, define ℓ^p to be the set of all real sequences $x = \{x_n, n \in \mathbb{N}\}$ satisfying

$$\|x\|_p = (\textstyle\sum_{n\in\mathbb{N}} |x_n|^p)^{1/p} < \infty.$$

Also define ℓ^∞ to be the set of all real sequences $x = \{x_n, n \in \mathbb{N}\}$ satisfying

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| < \infty.$$

Both ℓ^p and ℓ^∞ are normed vector spaces with respect to the norms $\|\cdot\|_p$ and $\|\cdot\|_\infty$ (do not prove this). Recall that a normed vector space is said to be separable if it contains a countable, dense set.

(a) For any $1 \leq p < \infty$, prove that ℓ^p is separable.

(b) Prove that ℓ^∞ is not separable. (**Hint.** Consider a proof by contradiction.)