BASIC EXAM – COMPLEX ANALYSIS

31 AUGUST 2005

Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

Notation. For $w \in \mathbb{C}$ and $r \in \mathbb{R}_{\geq 0}$, we let $B_r(w) = \{z \in \mathbb{C} : |z - w| < r\}$ be the open disc of radius r centered at w. We let $\mathbb{D} = B_1(0)$ be the open unit disc.

---

1. (a) Suppose f is an entire function such that $\text{Re}(f)$ is bounded. Prove that f is constant.

(b) Suppose f is an entire function such that $A\ \text{Re}(f) + B\ \text{Im}(f)$ is bounded, where A, B are real numbers at least one of which is not 0. Prove that f is constant.

---

2. Suppose $f : \mathbb{H} \to \mathbb{C}$ is an anlytic function on the upper half-plane $\mathbb{H}$ satisfying

$$|f(z)| \leq 1\ \forall z \in \mathbb{H}, \qquad \text{and } f(i) = 0.$$

Give a sharp upper bound for $|f(3i)|$. In other words, explicity give a constant M and prove that $|f(3i)| \leq M$, then find a function f satisfying the above hypotheses such that $|f(3i)| = M$.

---

3. Expand

$$\frac{1}{z(z-1)(z-2)}$$

in a Laurent series valid in the region:

(a) $0 < |z| < 1$

(b) $1 < |z| < 2$.

---

4. Show that there is no function f analytic on the punctured plane $\mathbb{C} - \{0\}$ that satisfies

$$|f(z)| \geq \frac{1}{\sqrt{|z|}} \qquad \forall z \neq 0.$$

---

5. Use complex analysis to evaluate

$$I_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin n\theta}{\sin \theta} d\theta$$

for every positive integer n.

---

6. Use complex analysis to evaluate

$$\int_0^\infty \frac{dx}{1+x^6}.$$

Justify all steps in the computation.

7. Suppose a polynomial $f(z) = \sum_{j=0}^n a_j z^j$ of degree n has zeros at $w_1, w_2, \dots, w_k$ of multiplicity $n_1, n_2, \dots, n_k$ respectively. Fix a real number ε satisfying

$$0 < \varepsilon < \min_{1 \le i < j \le k} \frac{|w_i - w_j|}{2}.$$

For $j = 1, \cdots, k$, let $B_j = B_\varepsilon(w_j)$ be the open disc of radius ε centered at w_j with boundary circle $\gamma_j = \{z : |z - w_j| = \varepsilon\}$. Prove that there is real number $\delta > 0$ such that every polynomial $g(z) = \sum_{i=0}^n b_i z^i$ satisfying

$$|a_i - b_i| \le \delta, \qquad 0 \le i \le n,$$

has n_j zeros in B_j for $j = 1, \dots, k$.

Hint: Apply Rouché's Theorem on each boundary circle γ_j.

8. Suppose $f : \mathbb{D} \to \mathbb{C}$ is analytic on $\mathbb{D}$ and satisfies

$$|f(z)| \le \frac{1}{1 - |z|}.$$

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be the Taylor expansion of f at 0. Prove that

$$|a_n| \le (n+1)\left(1 + \frac{1}{n}\right)^n < e(n+1) \qquad \forall n \ge 0.$$

9. Let $u : \Omega \to \mathbb{R}$ be a non-constant harmonic map on an open connected subset Ω of $\mathbb{C}$. Prove that u is an open mapping, i.e. it maps open sets to open sets.

10. Let $n \ge 1$ be an integer. Prove that the polynomial

$$e_n(z) = \sum_{j=0}^n \frac{z^j}{j!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}$$

has n distinct complex roots $z_1, z_2, \dots, z_n$ and that these satisfy

$$z_1^{-k} + z_2^{-k} + \cdots + z_n^{-k} = 0 \qquad \text{for } 2 \le k \le n.$$

Hint: For suitable integers ℓ, m, you may wish to consider either one of the following integrals:

$$I_\ell = \int_{C_r(0)} \frac{z^\ell}{e_n(z)} dz, \qquad \text{or} \qquad J_m = \int_{C_r(0)} \frac{z^m e_n'(z)}{e_n(z)} dz,$$

where $C_r(0)$ is the circle of radius r centered at 0.