DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. . Solve the problem

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad 0 < t < \infty$$

with boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0$$

and initial conditions

$$u(x,0) = 0 \text{ and } u_t(x,0) = \frac{1}{2} - \frac{\cos(2x)}{2}$$

2. Consider the problem $(d > 0,)$

$$(*) \begin{cases} u_t + du_{xxxx} = 0 \qquad , \quad x \in (0,1), t > 0 \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \\ u_x(0,t) = u_x(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = u_{xxx}(0,t) = u_{xxx}(1,t) = 0 \textit{ all } t > 0 \end{cases}$$

(a) Show that the energy

$$E(t) = \int_0^1 \frac{1}{2}(u_t^2 + du_{xx}^2)(x,t)dx$$

is constant in time

(b) Show that (*) has a unique solution **(Hint: use (a))**

3. Consider the Burgers equation

$$u_t + uu_x = 0 \quad , \quad x \in \mathbf{R}, \quad t > 0$$

with two different initial data, namely

$$u(x,0) = \varphi_1(x) = \begin{cases} 1 & x < 0 \\ 1 - x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

and

$$u(x,0) = \varphi_2(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

(a) Solve the equation using the method of characteristics for each one of the data.

(b) Discuss and compare the two solutions from part (a).

4. Assume $u_1 = u_2(x,t), u_2 = u_2(x,t)$ solve the diffusion equation

$$u_t = u_{xx} \quad , \quad x \in (0,1), \ t > 0$$

with data $u_1(x,0) = \varphi_1(x), \ \ u_2(x,0) = \varphi_2(x)$ respectively, where φ_1, φ_2 are smooth functions defined on $[0,1]$. Using the maximum principle, show that if

$$\varphi_1(x) \le \varphi_2(x) \quad , \quad x \in [0,1]$$

then

$$u_1(x,t) \le u_2(x,t) \quad \text{for all} \ \ x \in [0,1]\,, t > 0.$$

5. (a) Suppose M is an $n \times n$ matrix. $\lambda \in \mathbf{R}$ and $x \in \mathbf{R^n}$ are eigenvalues and eigenvectors of M, respectively. Let $\mu = \pm\sqrt{\lambda}$. Show that

$\begin{pmatrix} x \\ \mu x \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 0 & I \\ M & 0 \end{pmatrix}$ when I is the $n \times n$ identity matrix and 0 is the $n \times n$ zero matrix.

(b) Use (a) to find the general solution of

$$y' = \begin{pmatrix} 0 & I \\ M & 0 \end{pmatrix} y$$

where

$$M = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \textit{and } 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

6. Consider the system

$$\frac{dx}{dt} = -x + e^{-y}$$
$$\frac{dy}{dt} = -x - 2y + 1$$

(a) Use linearization to determine the behavior of solutions starting near the rest point $(x, y) = (1, 0)$.

(b) Give a reason why the system must possess a second rest point somewhere in the region $x > 1$. (You will not be able to solve for it algebraically).

(c) Sketch the vector field $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$ everywhere in the entire (x, y) plane and indicate in your drawing where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

(d) What does your drawing in c) suggest about the character of the second rest point in c)?

7. For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N: this is called the "Allee effect". For example, imagine that it is too hard to find mates when N is very small and

there is too much competition when N is large. Consider the population model

$$\dot{N}/N = r - a(N-b)^2$$

where r, a, and b are positive parameters.

(a) Find all fixed points of the system for $r, a, b > 0$.

(b) Describe the conditions on the parameters under which this model exhibits the Allee effect.

(c) Sketch the solutions $N(t)$ for all nonnegative initial conditions for a particular set of parameter values where the system exhibits the Allee effect. Does every system which admits the Allee effect behave essentially in this manner, or are there other possible behaviors that can arise?