Department of Mathematics and Statistics
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Basic Exam: Advanced Calculus/Linear Algebra

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Do 7 out of the following 9 problems. Indicate clearly which problems should be graded.

Passing standard: To pass at the Master's level it is sufficient to have 60% with **three** problems essentially complete (including at least one from each part). To pass at the Ph.D. level, 75% with **two questions from each part** essentially complete.

Part 1. Linear algebra

Problem 1. Find a 3×3 matrix A such that:

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \; ; \; A \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \; ; \; A \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Problem 2. Consider the antidiagonal matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find the eigenvalues of A.
b) Find an orthonormal basis of $\mathbb{R}^4$ consisting of eigenvectors of A.

Problem 3. Prove or disprove the following statements:

a) If a 2×2 complex matrix A is such that

$$\lim_{k \to \infty} A^k = I$$

then $A = I$.

b) If A is an $n \times n$ real non-singular matrix and $a \in \mathbb{R}$ is a non-zero eigenvalue of A, then there exists a (column) vector $v \in \mathbb{R}^n$ such that the matrix

$$B \;=\; A - a \cdot v \cdot v^T$$

satisfies $\text{rank}(B) < \text{rank}(A)$.

Problem 4. Let A be an $n \times n$ complex matrix such that $A^2 = A$.

a) Show that A is similar to a diagonal matrix.
b) Show that $tr(A)$ is a non-negative integer.

Part 2. Advanced Calculus

Problem 1. Let $\vec{F}$ be the vector field in $\mathbb{R}^3\backslash\{0\}$

$$\vec{F} \;=\; \frac{\vec{r}}{||\vec{r}||^3}$$

where $\vec{r} = (x, y, z)$. Let $a > 1$ and let X_a denote the surface

$$X_a = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{a^4} + \frac{z^2}{a^8} = 1\}$$

Compute the surface integral

$$\int_{X_a} \vec{F} \cdot dS\,.$$

Hint: Compare with the surface integral over the sphere of radius a centered at the origin.

Problem 2. Let $f_n(x)$ be a sequence of real valued functions

$$f_n \colon [0, 1] \to \mathbb{R}$$

which converge uniformly (on $[0, 1]$) to the zero function. Suppose, moreover, that

$$0 \le f_{n+1}(x) \le f_n(x)$$

for all n and all $x \in [0, 1]$. Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n f_n(x)$$

converges uniformly on $[0, 1]$.

Problem 3. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a C^2 function such that $f(0) = 0$ and $f'(x)$ is increasing for $x \ge 0$. Prove that $g(x) = f(x)/x$ is an increasing function for $x > 0$.

Problem 4. Let $f(x) = x^2 \displaystyle\int_0^x \cos(t^3)dt$. Compute $f^{(15)}(0)$ and $f^{(20)}(0)$, where $f^{(j)}(0)$ denotes the j-th derivative of f at 0.

Problem 5. Let $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map:

$$F(x, y) = (y^2 - x^2, xy)\,,$$

and $\Delta = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. Compute the area of the image $F(\Delta)$.