Do 7 out of the following 9 problems. Indicate clearly which problems should be graded.

Passing standard: To pass at the Master’s level it is sufficient to have 60% with three problems essentially complete (including at least one from each part). To pass at the Ph.D. level, 75% with two questions from each part essentially complete.

Part 1. Linear algebra

Problem 1. Find a $3 \times 3$ matrix $A$ such that:

\[
A \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix};
A \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix};
A \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.
\]

Problem 2. Consider the antidiagonal matrix

\[
A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]

a) Find the eigenvalues of $A$.
b) Find an orthonormal basis of $\mathbb{R}^4$ consisting of eigenvectors of $A$.

Problem 3. Prove or disprove the following statements:

a) If a $2 \times 2$ complex matrix $A$ is such that

\[
\lim_{k \to \infty} A^k = I
\]

then $A = I$.

b) If $A$ is an $n \times n$ real non-singular matrix and $a \in \mathbb{R}$ is a non-zero eigenvalue of $A$, then there exists a (column) vector $v \in \mathbb{R}^n$ such that the matrix

\[
B = A - a \cdot v \cdot v^T
\]

satisfies $\text{rank}(B) < \text{rank}(A)$.

Problem 4. Let $A$ be an $n \times n$ complex matrix such that $A^2 = A$.

a) Show that $A$ is similar to a diagonal matrix.
b) Show that $\text{tr}(A)$ is a non-negative integer.
Part 2. Advanced Calculus

Problem 1. Let $\vec{F}$ be the vector field in $\mathbb{R}^3 \setminus \{0\}$
\[ \vec{F} = \frac{\vec{r}}{||\vec{r}||^3} \]
where $\vec{r} = (x, y, z)$. Let $a > 1$ and let $X_a$ denote the surface
\[ X_a = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{a^4} + \frac{z^2}{a^8} = 1\} \]
Compute the surface integral
\[ \int_{X_a} \vec{F} \cdot dS. \]
**Hint:** Compare with the surface integral over the sphere of radius $a$ centered at the origin.

Problem 2. Let $f_n(x)$ be a sequence of real valued functions
\[ f_n : [0, 1] \to \mathbb{R} \]
which converge uniformly (on $[0, 1]$) to the zero function. Suppose, moreover, that
\[ 0 \leq f_{n+1}(x) \leq f_n(x) \]
for all $n$ and all $x \in [0, 1]$. Prove that the series
\[ \sum_{n=1}^{\infty} (-1)^n f_n(x) \]
converges uniformly on $[0, 1]$.

Problem 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2$ function such that $f(0) = 0$ and
$f'(x)$ is increasing for $x \geq 0$. Prove that $g(x) = f(x)/x$ is an increasing function for $x > 0$.

Problem 4. Let $f(x) = x^2 \int_0^x \cos(t^3) dt$. Compute $f^{(15)}(0)$ and
$f^{(20)}(0)$, where $f^{(j)}(0)$ denotes the $j$-th derivative of $f$ at 0.

Problem 5. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the map:
\[ F(x, y) = (y^2 - x^2, xy), \]
and $\Delta = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Compute the area of the image $F(\Delta)$. 