Part I.

1. Let $G$ be a finite group acting transitively on a set $S$. Let $H \trianglelefteq G$ be a normal subgroup of $G$. Let $O_1, \ldots, O_s$ be the orbits of $H$ acting on $S$.
   
   (a) Prove that these orbits all have the same cardinality.
   
   (b) Let $a \in S$ and let $G_a = \{g \in G \mid g.a = a\}$. Show that $s = |G : H G_a|$.

2. Let $G$ be a finite group and $P$ a non-trivial $p$-Sylow subgroup of $G$. Let $H$ be the normalizer of $P$ in $G$. Show that the normalizer of $H$ in $G$ is equal to $H$.

Part II. All rings are commutative with 1, and all ring homomorphisms take 1 to 1.

3. Let $k$ be a field. Recall that a $k$-algebra is a ring $R$ with a multiplicative identity $1_R$ and a map of rings $k \to R$ that takes the multiplicative identity of $k$ to $1_R$.
   
   (a) Let $V$ and $W$ be $k$-algebras. Show that $V \otimes_k W$ has a natural structure as a ring.
   
   (b) Show that the rings $\mathbb{C} \otimes_k \mathbb{C}$ and $\mathbb{C} \oplus \mathbb{C}$ are isomorphic rings.

4. Let $R$ be a commutative ring with identity. Recall that an element $p \in R$ is prime if the ideal generated by $p$ is a non-zero prime ideal.
   
   Show that if an element in an integral domain is expressible as a product of primes, then that expression is unique up to multiplication by units and permutations of the elements in the expression.

Part III.

5. Let $k$ be a field and $V$ a vector space of finite dimension $n$ over $k$. Let $A$ be a linear transformation of $V$ with minimal polynomial $(x - a)^n$ for some $a \in k$.
   
   (a) Find the Jordan canonical form of $A$.
   
   (b) Describe the set of linear maps from $V$ to itself which commute with $A$.

6. Let $M$ be a free $\mathbb{Z}$-module with basis $\{e_1, e_2, e_3\}$. Let $N$ be the submodule of $M$ generated by $\{e_1 - e_2 - e_3, e_1 - e_2 + e_3, -2e_1 + 10e_2 - 6e_3\}$.
   
   (a) Describe the isomorphism type of $N$ as a $\mathbb{Z}$-module.
   
   (b) Describe the isomorphism type of $M/N$ as a $\mathbb{Z}$-module.
   
   (c) Find a submodule $N'$ of $M$ such that $M = N + N'$ and the sum is a direct sum, or explain why no such $N'$ exists.
Part IV.

7. Let $f(x) = x^n - 1$.
   (a) Prove that the Galois group of $f(x)$ over the field of rational numbers is an abelian group.
   (b) Find the smallest $n$ such that the Galois group is not cyclic.

8. Let $K$ be a finite, separable extension of the field $k$. Prove that if $K$ is a splitting field over $k$ then every irreducible polynomial in $k[x]$ that has a root in $K$ splits completely in $K[x]$. 