DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - PROBABILITY
FRIDAY, SEPTEMBER 3, 2004

Work all problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. (20 points) Let X have a Poisson distribution.
   (a) Give the probability mass function for X.
   (b) **Derive** the moment generating function of X.
   (c) **Derive** the mean and variance of X. (You can do this using b) if you want, but you don't have to.)

2. (20 points) Suppose a plant is manufacturing a product using three different machines 1, 2 and 3 and large inventory has been built up which consists of 30% from 1, 20% from 2 and 50% from 3. Suppose 100 items are selected. With a large inventory we will treat the selections as independent where on each draw the probability is .3, .2 and .5 of getting an item from machine 1, 2 or 3, respectively.
   (a) **Derive** the joint distribution of X_1, X_2, X_3 where X_j = the number of itemsselected from machine j.
   (b) Suppose that each item has a lifetime, and the distribution of lifetimes has mean 5 and standard deviation 1 for machine 1; mean 6 and standard deviation .5 for machine 2; and mean 7 and standard deviation .8 for machine 3. Let T denote the total lifetime of the 100 items selected (see part a). Find the expected value and variance of T.

3. (20 points) Let $X = (X_1, X_2, \ldots X_n)$ be a vector valued random variable and let $T_i = T_i(X)$, $i = 1, 2, \ldots d$. Suppose that the probability density function of X, parameterized by $\eta = (\eta_1, \ldots, \eta_d)$ in an open interval in R^d, is given by

$$g_\eta(x) = h(x)\exp\{\sum_{i=1}^{d} \eta_i T_i(x) - K(\eta)\},$$

   where it is assumed that

$$\int_{R^d} h(x)\exp\{\sum_{i=1}^{d} \eta_i T_i(x)\}dx < +\infty.$$

   (a) Show that

$$K(\eta) = \log\Big(\int h(x)\exp\{\sum_{i=1}^{d} \eta_i T_i(x)\}dx\Big)$$

   (b) Show that

$$E[T_i] = \frac{\partial}{\partial \eta_i} K(\eta),$$

(c) Show that
$$\text{cov}(T_i, T_j) = \frac{\partial 2}{\partial \eta_i \partial \eta_j} K(\eta).$$

4. (25 points) Let X_1 and X_2 be independent exponential distributions with mean 1. Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1$.

   (a) Derive the joint distribution (giving the joint density suffices) of (Y_1, Y_2)

   (b) Find the conditional distribution of Y_1 given $Y_2 = y_2$.

   (c) Give an approximation to the variance of Y_1/Y_2.

5. (15 points) Let $(X_1, \ldots X_n)$ and $(Y_1, \ldots, Y_n)$ be two different i.i.d. random sequences with
$$E[X_i] = \mu_X, E[Y_i] = \mu_Y, \quad \text{Var}(X_i) = \sigma_X 2 > 0, \text{Var}(Y_i) = \sigma_Y 2 > 0.$$
   We denote
$$\bar{X} = n^{-1}(X_1 + \ldots + X_n), \quad \bar{Y} = n^{-1}(Y_1 + \ldots + Y_n).$$
   Identify *and justify* the limiting distribution of
$$\sqrt{n}(\bar{X} - \mu_X) + \bar{Y}$$
   as $(n \to +\infty)$.

   You can appeal to well known results but state clearly what results you are using and how they apply here.