DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

AUGUST 30, 2004

Passing Standard: It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the four parts.

Part I.

1. Let G be a finite group of order p^n, where p is a prime. Show that its center $Z(G)$ is non-trivial, and that G is solvable.

2. (a) Determine the number of *automorphisms* of the symmetric group S_3. **Show your work.**
(b) Determine the number of *homomorphisms* from S_3 to S_3. **Show your work.**

Part II.

All rings are commutative with 1, and all ring homomorphisms take 1 to 1.

1. Let A be a ring, and let $I, J \subset A$ be coprime ideals, i.e. $I + J = A$. For any positive integers m, n, show that I^m and J^n are also coprime ideals.

2. Determine all subrings of $\mathbf{Z} \times \mathbf{Z}$.

Part III.

All rings are commutative and with 1.

1. (a) Prove that there is no 3×3 matrix A over $\mathbf{Q}$ with $A^8 = I$ but $A^4 \neq I$.
(b) Write down a 4×4 matrix B over $\mathbf{Q}$ with $B^8 = I$ but $B^4 \neq I$. (NOTE: you must show that B satisfies these conditions)

2. Let $f, g \in \mathbf{C}[x]$ be non-constant polynomials. Prove or disprove:
(a) $\mathbf{C}[x]/f \otimes_{\mathbf{C}[x]} \mathbf{C}[x]/g \simeq \mathbf{C}[x]/\gcd(f, g)$ as $\mathbf{C}[x]$-modules.
(b) $\mathbf{C}[x]/f \otimes_{\mathbf{C}} \mathbf{C}[x]/g \simeq \mathbf{C}[x]/\gcd(f, g)$ as $\mathbf{C}$-modules.

Part IV.

1. Let K/k be a Galois extension of degree 45.
(a) does there always exist a field F with $k \subset F \subset K$ such that $[F : k] = 5$?
(b) does there always exist F as in (a) *and* with F/k Galois?

2. Denote by $\mathbf{F}_2$ the finite field with two elements.
(a) Show that $g(x) = x^4 + x^3 + x^2 + x + 1$ is irreducible and separable over $\mathbf{F}_2$.
(b) Denote by K a splitting field of g over $\mathbf{F}_2$, and let $\alpha \in K$ be a root of g. Show that $K = \mathbf{F}_2(\alpha)$.
(c) Determine $\mathrm{Gal}(K/\mathbf{F}_2)$.
