Department of Mathematics and Statistics
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Basic Exam: Topology

August 29, 2003

Answer five of seven questions. Indicate clearly which five questions you want to have graded. Justify your answers.

(1) Show that any continuous map $f\colon [0,1] \to [0,1]$ has a fixed point, i.e. there is a point $x \in [0,1]$ such that $f(x) = x$. What happens if $[0,1]$ is replaced by $(0,1)$? Justify your answer.

(2) Let $f\colon X \to Y$ be a continuous bijection from a compact space X to a Hausdorff space Y. Show that f is a homeomorphism.

(3) Let $\mathbb{R}$ be the real numbers with the usual topology, and let $\mathbb{R}_l$ be the same set with the topology having all half open intervals $[a,b)$, $a < b$ as a basis. Prove or disprove: all continuous functions $f\colon \mathbb{R} \to \mathbb{R}_l$ are constant.

(4) Prove that no two of the following spaces are homeomorphic: $\mathbb{R}$, $\mathbb{R}^2$ (both with the standard topology), and $\mathbb{R}^\omega$, the product of countably many copies of $\mathbb{R}$, with the product topology.

(5) Let (X, d_X), (Y, d_Y) be metric spaces, and suppose that Y is complete and $A \subset X$ is dense. Show that if $f\colon A \to Y$ is uniformly continuous, there exists a continuous function $\overline{f}\colon X \to Y$ extending f, and that it is unique.

Show by example that it is not enough for f to be continuous.

(6) Suppose that $A \subset \mathbb{R}^2$ is compact, and $B \subset \mathbb{R}^2$ is closed. Show that the set

$$C = \{p \in \mathbb{R}^2 \mid (p + A) \cap B = \emptyset\}$$

is open. (here we put $p + A = \{p + a \mid a \in A\}$)

(7) Let $\mathcal{C}$ be the set of all continuous functions $[0,1] \to [0,1]$, with the compact-open topology.
  (a) Is $\mathcal{C}$ compact? Justify your answer.
  (b) Show that the set

$$\{(f, x) \in \mathcal{C} \times [0,1] \mid f(x) = x\}$$

  is closed in $\mathcal{C} \times [0,1]$ (with the product topology).