Do 5 out of the following 7 questions. Indicate clearly what questions you want to have graded.
Passing standard: 70% with three problems essentially complete. Justify all your answers.

Problem 1. Prove or disprove: a simply connected manifold is orientable.

Problem 2. Let $f : M \to \mathbb{R}$ be a smooth, proper function on a connected surface $M$. (Recall that $f$ is proper iff the preimage of any compact set under $f$ is compact.)

(1) What is the preimage of a regular value of $f$? (Hint: Use the classification of 1-manifolds.)
(2) Show that if $f$ has no critical points, then $M$ is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.

Problem 3. Consider the vector fields $X = y\partial_z - z\partial_y$, $Y = z\partial_x - x\partial_z$, $Z = x\partial_y - y\partial_x$ on $\mathbb{R}^3$.

(1) Describe the integral curves of $X$, $Y$ and $Z$, as well as the corresponding flows on $\mathbb{R}^3$.
(2) Show that the span of $X$, $Y$ and $Z$ defines a rank 2 subbundle $E \subset T_N$, where $N = \mathbb{R}^3 \setminus \{0\}$, and find a 1-form $\alpha$ on $N$ whose kernel is $E$.
(3) Prove that $N$ is foliated by leaves (integral manifolds) tangent to $E$; please compute these leaves explicitly.

Problem 4. Suppose $L$ is a real line bundle over a compact manifold $M$. Which of the following vector bundles over $M$ necessarily has a global non-vanishing section (proof or counterexample):

(1) The bundle $L$ itself?
(2) The tensor product $L \otimes L$?
(3) The direct sum $L \oplus L$?
(4) The rank $n$ bundle $L \oplus \cdots \oplus L$ for $n \geq 3$?

Problem 5. Consider the infinite dimensional vector space $\Omega^2(T^4)$ of all smooth 2-forms on the flat 4-torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$.

(1) Verify that the Hodge operator * on $\Omega^2(T^4)$ satisfies $*^2 = 1$, and thus there is a decomposition $\Omega^2(T^4) = \Omega_+ \oplus \Omega_-$ into ±1-eigenspaces, the self-dual and anti-self-dual 2-forms on $T^4$.
(2) Show that the harmonic 2-forms on $T^4$ comprise a vector subspace $H \subset \Omega^2(T^4)$ isomorphic to the constant-coefficient 2-forms on $\mathbb{R}^4$. (Recall that a form $\omega$ is harmonic iff $d\omega = 0 = d^* \omega$.)
(3) Find a basis for $H$ consisting of self-dual and anti-self-dual 2-forms.

Problem 6. The graph $z = f(x, y)$ of the function $f(x, y) = xy$ defines a smooth surface $\Sigma \subset \mathbb{R}^3$ in Euclidean space.

(1) Determine the induced Riemannian metric on $\Sigma$ and show that it is complete.
(2) Compute the Gauss and mean curvatures of $\Sigma$ at the origin.
(3) Describe parallel translation in $\Sigma$ along the curve $\{x = 0\}$.

Problem 7. Let $G \subset GL(2, \mathbb{R})$ be the set of all 2-by-2 matrices $A$ such that $A^tQA = Q$, where $Q$ is the diagonal matrix with entries 1 and −1.

(1) Show that $G$ is a Lie group, determine its Lie algebra and calculate its dimension.
(2) How many components does $G$ have?
(3) Give an explicit parametrization of the identity component of $G$ via the exponential map.