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Advanced Analysis Qualifying Exam
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Instructions.

(1) This exam consists of eight (8) problems all counted equally for a total of 100%.

(2) You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

(3) In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

(4) State explicitly all results that you use in your proofs and verify that these results apply.

(5) Please write your work and answers clearly in the blank space under each question.

Conventions.

(1) For a set A, 1_A denotes the indicator function or characteristic function of A.

(2) If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

(3) If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1) Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let u and v be linearly independent *unit* vectors in $\mathcal{H}$. Define $\mathcal{M}$ to be the span of u and v.

(a) Determine a *unit* vector w such that $\langle u, w \rangle = 0$ and the span of u and w equals $\mathcal{M}$ (make sure to verify the latter!).

(b) Let x be an element in $\mathcal{H} \setminus \mathcal{M}$. Determine explicitly, in terms of u and w, $y_0 \in \mathcal{M}$ such that

$$\|x - y_0\| = \inf\{\|x - z\| : z \in \mathcal{M}\}.$$

(c) Prove that the y_0 found in (b) is unique and re-express it in terms of u and v.

2) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $f \in L^1(X)$, $f \geq 0$. Let $\lambda_f : (0, \infty) \to [0, \infty]$ be the distribution function of f; i.e.

$$\lambda_f(t) := \mu(\{x \in X : f(x) > t\})$$

(a) Prove that $\lambda_f(t) < \infty$ for all $t > 0$.

(b) Show that λ_f is decreasing and right continuous..

(c) In light of (b), the distribution function λ_f defines a *negative* Borel measure ν on $(0, \infty)$ by

$$\nu(\,(a, b]\,) := \lambda_f(b) - \lambda_f(a) \quad \text{whenever} \quad 0 < a < b.$$

Moreover,

$$\int \phi d\lambda_f = \int \phi d\nu$$

is the Lebesgue -Stieljes integral of functions ϕ defined on $(0, \infty)$.

Consider ψ a nonnegative Borel measurable function on $(0, \infty)$ function and f as above. Prove the following equality

$$\int_X \psi \circ f \, d\mu \;=\; -\int_0^\infty \psi(t)\, d\lambda_f(t).$$

Hint. Prove it first when ψ is the characteristic of a Borel set and then suitably approximate ψ by simple functions.

3) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ and let $\mathcal{B}_2 = \mathcal{B} \otimes \mathcal{B}$ be the product σ-algebra on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Using the definition of product σ-algebras show that the open unit disc

$$\mathcal{D} := \{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 < 1\}$$

belongs to $\mathcal{B}_2$.

4) Let $f \in L^1(\mathbb{R}, m)$ and $r > 0$. Set

$$A_r(f)(x) := \frac{1}{2r} \int_{x-r}^{x+r} f(y)\, dy.$$

(a) Show that $A_r(f)(x)$ is continuous in *both* x and r.

(b) Show that if in addition f is continuous, then

$$\lim_{r \to 0} A_r(f)(x) \;=\; f(x).$$

(c) Show that A_r is a contraction in $L^1(\mathbb{R})$ in the sense that

$$\|A_r(f)\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}.$$

(d) Use (b) and (c) to show that if $f \in L^1(\mathbb{R})$ then

$$\lim_{r \to 0} \|A_r(f) - f\|_{L^1(\mathbb{R})} \;=\; 0.$$

Hint. You may use without proof the fact that $L^1(\mathbb{R})$ functions can be approximated in $L^1(\mathbb{R})$ by continuous functions.

5) Let μ be a positive Borel measure on X and let $f : X \to [0,\infty)$ be a measurable function such that

$$\int_X f\, d\mu = M\,, \qquad 0 < M < \infty.$$

Compute

$$\lim_{n\to\infty} \int_X n\, \log\left[1 + \left(\frac{f(x)}{n}\right)^{\alpha}\right] d\mu$$

for constant $\alpha > 0$ in the following cases.

(a) $0 < \alpha < 1$

(b) $\alpha = 1$

(c) $1 < \alpha < \infty$

Hint. Use Fatou in (a).

6) Let $f \in L^1(\mathbb{R}, m)$ and let $\hat{f}$ be its Fourier transform defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x)\, e^{-i\xi x}\, dx$$

for all $\xi \in \mathbb{R}$.

(a) Prove that $\hat{f}$ is uniformly continuous on $\mathbb{R}$.

(b) Assume $f \in L^1(\mathbb{R})$ satisfies

$$\int_{|x|\le N} |x|\, |f(x)|\, dx \le N^{1/2}$$
$$\int_{|x|>N} |f(x)|\, dx \le \frac{1}{N^{1/2}}$$

for all $0 < N < \infty$.

Show then that $\hat{f}(\xi)$ is Hölder continuous; that is, show that for all ξ and $\eta \in \mathbb{R}$

$$|\hat{f}(\xi) - \hat{f}(\eta)| \le C\, |\xi - \eta|^{\beta}$$

for some suitable positive constants C and β which are independent of ξ and η.

Hint. Obtain an estimate for $\int_{|x|\le N} (e^{-ix\xi} - e^{-ix\eta})\, f(x)\, dx$ and another one for $\int_{|x|>N} (e^{-ix\xi} - e^{-ix\eta})\, f(x)\, dx$. Then optimize your choice of N to obtain the desired estimate.

(c) Now assume that in addition to having $f \in L^1(\mathbb{R}, m)$, the function $xf(x)$ is also integrable; that is, $\int_{\mathbb{R}} |xf(x)|\, dx < \infty$.

Show then that $\hat{f}$ is differentiable and moreover that

$$\frac{d}{d\xi}\hat{f}(\xi) = \widehat{(-ixf)}(\xi).$$

Hint. Use the Dominated Convergence Theorem.

7) Let $f : [0,1] \to [0,\infty)$ be a Lebesgue measurable function such that

$$\int_0^1 e^{[f(x)]}\, dx < \infty.$$

Define $F(s) := m(f^{-1}([s,\infty)) = m(\{x \in [0,1] : f(x) \geq s\})$. Show that

$$\lim_{s\to\infty} e^s\, F(s) = 0.$$

8) Let f and $\{f_n\}_{n\geq 1}$ be real-valued functions on the unit interval $[0,1]$ which are measurable with respect to Lebesgue measure.

(a) Suppose that $f_n(x) \to f(x)$ for all x in $[0,1]$. Show then that $\{f_n\}$ converges in measure to f. That is, show that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} m(\{x \in [0,1] \,:\, |f_n(x) - f(x)| > \varepsilon\} \;=\; 0.$$

(b) Suppose that $\{f_n\}$ converges in measure to f. Show that

$$\lim_{n\to\infty} \int_0^1 \cos(\, f_n(x)\,)\, dx \;=\; \int_0^1 \cos(\, f(x)\,)\, dx.$$