Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

August 28, 2002

Answer five of seven questions. Indicate clearly which five questions you want to have graded. Justify your answers.

Passing Standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

1. Let $f : X \to Y$ be a continuous map, with graph $\Gamma_f \subset X \times Y$.

   (a) If X is connected, prove that Γ_f is connected.

   (b) If Y is Hausdorff (which means the diagonal $\Delta \subset Y \times Y$ is closed), prove that Γ_f is closed.

2. Consider $\mathbb{R}$ with the standard topology as well as $\mathbb{R}_\ell$: the real numbers with the lower limit topology, whose basis consists of the intervals $[a, b)$.

   (a) Determine all continuous maps $f : \mathbb{R} \to \mathbb{R}_\ell$.

   (b) Determine all continuous maps $f : \mathbb{R}_\ell \to \mathbb{R}$.

3. Prove that $\mathbb{Q}$ (in the subspace topology of $\mathbb{R}$) is (a) totally disconnected, (b) not locally compact.

4. (a) Let $X = \mathbb{R}$ (standard topology), with the equivalence relation $x \sim y$ iff $x - y \in \mathbb{Z}$. Prove that the quotient space $X/\sim$ is homeomorphic to the unit circle $S^1 \subset \mathbb{R}^2$.

   (b) Let $X = \mathbb{R}^2$ (standard topology), with the equivalence relation $(x_1, x_2) \sim (y_1, y_2)$ iff $x_1 - y_1 \in \mathbb{Z}$ and $x_2 - y_2 \in \mathbb{Z}$. Prove that $X/\sim$ is compact Hausdorff.

5. Let X be a compact metric space, with metric d. Suppose the points $x_1, x_2, \dots \in X$ satisfy $d(x_n, x_m) \geq \varepsilon$ for all $n \neq m$. Prove that $\{x_n\}$ must be finite.

6. For a space X, the set $\mathcal{C}(X, \mathbb{R})$ of continuous $\mathbb{R}$-valued functions has two topologies: point-open (topology of pointwise convergence), compact-open. For a subspace Z of X, prove that the restriction map

$$r : \mathcal{C}(X, \mathbb{R}) \to \mathcal{C}(Z, \mathbb{R})$$

   is continuous in each of these topologies.

7. Let $\mathbb{R}^\omega$ be the set of all sequences of real numbers (a countable product of copies of $\mathbb{R}$), with the product topology, and let $\mathbb{R}^\infty$ be the subspace consisting of sequences $x = (x_n)$ such that $x_n = 0$ for sufficiently large n (depending on x).

   (a) Prove that $\mathbb{R}^\infty$ is dense in $\mathbb{R}^\omega$.

   (b) Find a countable dense subset of $\mathbb{R}^\omega$.

   (c) Prove that $\mathbb{R}^\omega$ has a countable basis.