Answer five of seven questions. Indicate clearly which five questions you want to have graded. Justify your answers.

Passing Standard: For Master’s level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

1. Let \( f : X \to Y \) be a continuous map, with \( \Gamma_f \subset X \times Y \).
   (a) If \( X \) is connected, prove that \( \Gamma_f \) is connected.
   (b) If \( Y \) is Hausdorff (which means the diagonal \( \Delta \subset Y \times Y \) is closed), prove that \( \Gamma_f \) is closed.

2. Consider \( \mathbb{R} \) with the standard topology as well as \( \mathbb{R}_\ell \): the real numbers with the lower limit topology, whose basis consists of the intervals \( [a, b) \).
   (a) Determine all continuous maps \( f : \mathbb{R} \to \mathbb{R}_\ell \).
   (b) Determine all continuous maps \( f : \mathbb{R}_\ell \to \mathbb{R} \).

3. Prove that \( \mathbb{Q} \) (in the subspace topology of \( \mathbb{R} \)) is (a) totally disconnected, (b) not locally compact.

4. (a) Let \( X = \mathbb{R} \) (standard topology), with the equivalence relation \( x \sim y \iff x - y \in \mathbb{Z} \). Prove that the quotient space \( X/\sim \) is homeomorphic to the unit circle \( S^1 \subset \mathbb{R}^2 \).
   (b) Let \( X = \mathbb{R}^2 \) (standard topology), with the equivalence relation \( (x_1, x_2) \sim (y_1, y_2) \iff x_1 - y_1 \in \mathbb{Z} \) and \( x_2 - y_2 \in \mathbb{Z} \). Prove that \( X/\sim \) is compact Hausdorff.

5. Let \( X \) be a compact metric space, with metric \( d \). Suppose the points \( x_1, x_2, \ldots \in X \) satisfy \( d(x_n, x_m) \geq \varepsilon \) for all \( n \neq m \). Prove that \( \{x_n\} \) must be finite.

6. For a space \( X \), the set \( C(X, \mathbb{R}) \) of continuous \( \mathbb{R} \)-valued functions has two topologies: point-open (topology of pointwise convergence), compact-open. For a subspace \( Z \) of \( X \), prove that the restriction map

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r : C(X, \mathbb{R}) \to C(Z, \mathbb{R})
\]

is continuous in each of these topologies.
7. Let $\mathbb{R}^\omega$ be the set of all sequences of real numbers (a countable product of copies of $\mathbb{R}$), with the product topology, and let $\mathbb{R}^\infty$ be the subspace consisting of sequences $x = (x_n)$ such that $x_n = 0$ for sufficiently large $n$ (depending on $x$).

(a) Prove that $\mathbb{R}^\infty$ is dense in $\mathbb{R}^\omega$.
(b) Find a countable dense subset of $\mathbb{R}^\omega$.
(c) Prove that $\mathbb{R}^\omega$ has a countable basis.