DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM – PROBABILITY
August 28, 2002

Work all problems. 60 points are needed to pass at the Master's level and 75 to pass at the Ph.D. level.

1. (20 pts) Choose a point P in the plane by letting the "x" and "y" coordinates, P_x and P_y, be independent $N(0,1)$ random variables.

   (a) What is the joint density function, $f(x,y)$, of P_x and P_y?

   (b) Create the point Q by rotating P clockwise by θ radians around the origin. Express the coordinates of Q, Q_x, Q_y as a function of the coordinates of P and find the joint density function for Q_x and Q_y.

   (c) Are Q_x and Q_y independent?

2. (20 pts) The number of eggs Y laid by an insect has a Poisson distribution with expected value $E(Y) = \lambda$. Given the insect lays $Y = y$ eggs, the number of those surviving, X, has a Binomial distribution with sample size y and probability p.

   (a) Compute the overall expected number of eggs that will survive. You must justify your answer.

   (b) Show that $V(X) = E(X)$.

   (c) Suppose that the average number λ of eggs laid by an insect is a function of the age β. In particular, assume that λ has an exponential distribution with parameter $\beta = E(\lambda)$. Compute the overall expected number of eggs that will survive as a function of β. You must justify your answer.

3. (20 pts) A rat is exposed to a known dose of X units of poison and either survives ($Y = 1$) or dies ($Y = 0$). Suppose the rat also has an unobserved natural tolerance to the poison (Z), and assume that this tolerance has a standard normal distribution. Further, suppose that the rat survives if and only if $Z > -X$. Note that X is a fixed quantity, and Z is random.

   (a) What is the probability that the rat survives?

   (b) What is the distribution of Z given that $Y = 1$?

   (c) Derive the moment generating function for Z given that $Y = 1$. You might want to express you answer in terms of Φ or ϕ where $\phi(\cdot)$ is the standard Gaussian PDF and $\Phi(\cdot)$ is the standard Gaussian CDF.

   (d) Again, assume that the rat survives. Use the moment generating function derived in part (c) to show that

$$E(Z|Y = 1) = \frac{\phi(-X)}{1 - \Phi(-X)} = \frac{\phi(X)}{\Phi(X)}$$

4. (25 pts) Suppose a person is at risk for two ways of dying, dying from cancer or dying from a heart attack. The time until a heart attack is modeled with an exponential distribution with mean μ^{-1} months, and the time until death from cancer is modeled with an exponential distribution with mean λ^{-1} months. Let the two times be independent. We will only observe the time until death (the minimum of the two exponential random variables) and the cause of death. The larger of the two random variables will not be observed.

   (a) What is the distribution of the observed time until death?

   (b) Find the probability that the person dies from cancer.

   (c) Show that the probability that the person dies of cancer within k months is $\frac{\lambda}{\mu+\lambda}\{1 - e^{-k(\mu+\lambda)}\}$.

   (d) Use parts (a), (b), and (c) to determine whether the time until death is independent of the cause of death.

5. (15 pts) Let $X_1, X_2, \dots$ be a sequence of independent random variables, with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$, and define

$$Y_n = \frac{\sum_{i=1}^n (X_i - \mu_i)}{(\sum_{i=1}^n \sigma_i^2)^{1/2}}.$$

It can be shown that, under suitable conditions,

$$\lim_{n\to\infty} P(Y_n \leq y) = \Phi(y)$$

where $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution.

Suppose that a computer test can generate an infinite number of questions arranged in a sequence from the easiest to the most difficult. Suppose also that the probability that a student will answer the ith question correctly is

$$p_i = \frac{1}{i+1}$$

and that all questions will be answered independently. Approximate the probability that a student will answer correctly at least 10 out of 100 questions.