

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - NUMERICS
August, 2002

Do five of the following problems. All problems carry equal weight.

Passing level:

Masters: 60% with at least two substantially correct.

Ph.D.: 75% with at least three substantially correct.

1. Prove that the iteration

$$x_{n+1} = e^{-x_n}$$

converges for all starting values x_0 .

2. (a) Write down the Jacobi and the Gauss-Seidel methods for the system $Ax = b$ where

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

- (b) Do these methods converge? Prove or disprove.

3. Find the best quadrature approximation of the form

$$I_h(f) = \alpha f(0) + \beta f(2h)$$

for the integral

$$I(f) = \int_0^{3h} f(x) dx,$$

and give the error term.

4. Consider the method

$$y_{i+2} = 4 y_{i+1} - 3 y_i - 2 h f(t_i, y_i)$$

for solving the ODE

$$\frac{dy}{dt} = f(t, y).$$

- (a) Find the local truncation error of the method.
- (b) Discuss the stability of the method.

5. Find the polynomial $p(x)$ of least order for which

$$\begin{aligned} p(x_1) &= y_1, & p(x_2) &= y_2, \\ p'(x_1) &= y'_1, & p'(x_2) &= y'_2, \end{aligned}$$

where $x_1 \neq x_2$ and y_i and y'_i are constants.

6. Suppose that you are to make a table of values of $\sin(x)$, $0 \leq x \leq \frac{\pi}{2}$, with a stepsize h . Assume linear interpolation on each subinterval ($I_i = [x_{i-1}, x_i]$, $i = 1, \dots, n$ with $x_0 = 0$ and $x_n = \frac{\pi}{2}$) is to be used with the table, and suppose the total error, including the effects due to rounding in the table entries, is at most 10^{-6} . What should h equal and to how many significant digits should the table entries be given?

7. Find the value of α that minimizes

$$\int_0^1 |e^x - \alpha| dx.$$

What is the minimum value?