Department of Mathematics and Statistics
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# Basic Exam - Complex Analysis

August 2002

**Do eight out of the following 10 questions.** Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

**Note:** All answers should be justified.

1. Give clear and precise statements of
   (a) Riemann's mapping theorem
   (b) Riemann's theorem on removable singularities.
   (c) Schwarz's lemma.

2. (a) Determine the number of zeroes of
$$z^4 - z^3 + 2z^2 + 3z + 5$$
   in the disk $\{z \ : \ |z| < 3\}$.
   (b) Evaluate the integral
$$\int_C \frac{2z^3 + z + 1}{z^4 - z^3 + 2z^2 + 3z + 5}\, dz$$
   where $C$ is the circle $\{|z| = 3\}$ transversed counterclockwise.
   [Hint: you may change the variable of integration to $w = 1/z$.]

3. Evaluate the integral
$$\int_0^{+\infty} \frac{x\sin(x)}{x^4 - \pi^4}\, dx.$$
   Justify your answer. Include the path of integration, and precise statements of all estimates involved in the justification, that the integral you computed is equal to the improper integral above. You may omit the proof of the estimates.

4. Evaluate the integral
$$\int_0^{\pi} \frac{1}{5 - 4\cos(\theta)}\, d\theta.$$
   Justify all steps.

5. (a) Find a one-to-one conformal map from the open set

$$D_1 \;:=\; \{z\;;\; |z-i|>1 \text{ and } \mathrm{Im}(z)>0\}$$

($D_1$ is the the complement of a closed disk in the upper half plane) onto the strip

$$D_2 \;:=\; \{z\;;\; 0<\mathrm{Im}(z)<\pi\}.$$

(b) Find a one-to-one conformal map from $D_1$ onto the upper half plane $\mathbb{H} := \{z\;;\; \mathrm{Im}(z)>0\}$.

6. (a) Find a function $u$, harmonic on the domain $\Omega := \{|z|<R \text{ and } Im(z)>0\}$, with the following boundary values:

$$\begin{cases} u=0 & \text{on } \{|z|<R \text{ and } Im(z)=0\}, \\ u=1 & \text{on } \{|z|=R \text{ and } Im(z)>0\}. \end{cases}$$

(b) Find the level sets of $u$.

7. Consider the Laurent series

$$\frac{1}{\sin(z)} \;=\; \sum_{n=-\infty}^{\infty} a_n z^n$$

valid for $\pi<|z|<2\pi$. Find the coefficients $a_{-1}$ and $a_{-2}$.

8. Let $g(w)$ be an analytic function on the complex plane except possibly at the origin, and such that

(a) $|wg(w)|\le 1$,

(b) $g(3)=\frac{1}{3}$,

(c) $\lim_{w\to\infty} g(w)=0$

Calculate $g(1)$. Prove your answer.

9. (a) Find the domain of analyticity of the series

$$f(z) \;=\; \sum_{n=1}^{+\infty} \frac{(1-z^{-1})^n}{n}.$$

(b) Show that $f'(z)$ extends to a meromorphic function on $\mathbb{C}$.

10. Classify the singularity of $f(z) = (z-2)\,\sin(\frac{1}{z+2})$ at $z=-2$; i.e., determine whether it is removable, a pole, or an essential singularity. Prove your answer.