University of Massachusetts
Department of Mathematics and Statistics
Advanced Exams in Geometry
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Do 5 out of the following 7 questions. Indicate clearly what questions you want to have graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

Problem 1. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be a 2-torus and consider the trivial rank n bundle $V = M \times \mathbb{R}^n$ over M. Denote by d the trivial connection on V (directional derivative of $\mathbb{R}^n$-valued functions). Let $\nabla = d + Adx + Bdy$ be a connection on V where A, B are real $n \times n$ matrices and dx, dy are the coordinate differentials on $\mathbb{R}^2$ which are well defined closed (but not exact) 1-forms on M. Show :

(1) ∇ is flat if and only if the matrices A and B commute, i.e., $[A, B] = 0$.
(2) Assuming ∇ to be flat, calculate the holonomy representation $H : \mathbb{Z}^2 \to \mathbf{Gl}(n, \mathbb{R})$.
(3) Assuming ∇ to be flat then ∇ admits a non-trivial parallel section if and only if A and B have a common kernel.

Problem 2. Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and define $\mathbf{Sp}(n) = \{A \in \mathbf{Gl}(2n, \mathbb{R})\,,\ AJA^T = J\,,\ \det A = 1\}$. Show that $\mathbf{Sp}(n)$ is a Lie group and determine its dimension and Lie algebra.

Problem 3. Let M be a compact manifold.

(1) Explain what is meant by a volume form on M.
(2) If M is $2n$-dimensional we call M *symplectic* if there exist a closed 2-form $\omega \in \Omega^2(M, \mathbb{R})$, i.e., $d\omega = 0$, so that $\omega \wedge \cdots \wedge \omega$ (n-times) is a volume form on M. Show that S^{2n} is *not* symplectic for $n > 1$.

Problem 4.

(1) Let F_1 and F_2 be homogeneous polynomials in the variables $x_0, \ldots, x_n$, of degree d_1 and d_2, respectively. Suppose moreover that the matrix $(\frac{\partial F_i}{\partial x_j})_{i,j}$ has rank 2 everywhere in $\mathbb{R}^{n+1} \backslash \{0\}$. Prove that the common zero set
$$M = \{[x_0 : x_1, \cdots : x_n] \in \mathbb{RP}^n \ :\ F_1(x) = F_2(x) = 0\}$$
is a smooth submanifold of $\mathbb{RP}^n$.
(2) Let $F_1(x_0, \ldots, x_3) = x_0x_3 - x_1x_2$, $F_2(x_0, \ldots, x_3) = x_1^2 - x_0x_2$. Prove that
$$M = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{RP}^3 \ :\ F_1(x) = F_2(x) = 0\}$$
has a unique singular point P. That is, M is not smooth but there exists a point $P \in \mathbb{RP}^3$ such that $M \backslash \{P\}$ is a smooth submanifold of $\mathbb{RP}^3$. What is the dimension of $M \backslash \{P\}$? Describe M.
(3) Let $F_3(x_0, \ldots, x_3) = x_2^2 - x_1x_3$ and F_1, F_2 as above. Prove that
$$M' = \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{RP}^3 \ :\ F_1(x) = F_2(x) = F_3(x) = 0\}$$
is a smooth submanifold of $\mathbb{RP}^3$. What is the dimension of M'? Describe M'.

Problem 5. Let $\pi : V \to \mathbb{RP}^n$ be the tautological line bundle whose fiber over $[x] \in \mathbb{RP}^n$ is given by the line

$$V_{[x]} = \mathbb{R}x \subset \mathbb{R}^{n+1}\,.$$

As usual, we view points in $\mathbb{RP}^n$ as equivalence classes in $\mathbb{R}^{n+1} \setminus \{0\}$.

(1) Let $U_i = \{[x] \in \mathbb{RP}^n : x_i \neq 0\}$. Show that V is trivial over U_i.
(2) Compute the transition functions g_{ij} relative to the covering $\{U_i\}$, $i = 0, \ldots, n$.
(3) Find all the global sections of the dual bundle V^*.

Problem 6.
Let $M^{n-1} \subset \mathbb{R}^n$ be a hypersurface and denote by $V \to M$ its normal line bundle. Show that V is trivial if and only if M is orientable. What can you say for a hypersurface in a non-orientable manifold?

Problem 7. Let C be the connected circular cone in $\mathbb{R}^3$ of opening angle α without its vertex.

(1) Find a domain $D \subset \mathbb{R}^2$ and an immersion $f : D \to \mathbb{R}^3$ so that $f(D) = C \setminus L$ where L is one of the generating lines of the cone and the induced metric $< df, df >= dx^2 + dy^2$ is the standard flat metric on D.
(2) Find all the geodesics on C.
(3) Show that the Levi-Civita connection on C is flat and parallel transport is path dependent. In contrast, parallel transport on D is path independent.