DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

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Passing Standard: It is sufficient to do FIVE problems correctly, including at least ONE FROM EACH of the four parts.

Part I.

1. Show that every group of order 175 is Abelian.

2. Denote by D_n the dihedral group of order $2n$ (the symmetry group of a regular n-gon), and by S_n the symmetric group on n letters. Count the total number of homomorphisms from S_4 to D_6. You can use the fact that $\#\mathrm{Aut}(D_3) = 6$.

Part II.

1. Let $A = \mathbf{Q}[x]/(x^2\mathbf{Q}[x])$.
(a) Find a basis for A as a vector space over $\mathbf{Q}$. Justify.
(b) Which elements of A are units?
(c) List all the ideals of A.
(d) Determine every ring homomorphism from A to $\mathbf{C}$ (the field of complex numbers) that sends $1 \in A$ to $1 \in \mathbf{C}$.

2. Let A be a commutative ring with 1. An element $x \in A$ is called *nilpotent* if $x^n = 0$ for some integer $n > 0$.
(a) Show that the set N of nilpotent elements of A is an ideal of A.
(b) Assuming (a), show that the quotient ring A/N contains no nonzero nilpotent element.
(c) Determine N for $A = \mathbf{Z}/60$.

Part III.

1. Let R be a principal ideal domain. For any R-module M, denote by M_{tor} the torsion submodule of M, and define $M_f := M/M_{\mathrm{tor}}$.
Let A, B, C be finitely generated R-modules for which $A \otimes_R B \simeq C \otimes_R B$ as R-modules.
(a) Prove or disprove: $A_f \simeq C_f$ as R-modules.
(b) Prove or disprove: $A_{\mathrm{tor}} \simeq C_{\mathrm{tor}}$ as R-modules.

2. Determine the number of conjugacy classes in the group $GL_2(\mathbf{F}_p)$ of 2×2 matrices over $\mathbf{F}_p$, the finite field with p elements (p prime). (Hint: consider the possible invariant factor polynomials).

Part IV.

1. Fix an element α in the finite field $\mathbf{F}_q$, where $q = p^n$ for some prime p. If α is not a p-power in $\mathbf{F}_q$, show that the polynomial $x^p - \alpha$ is irreducible in $\mathbf{F}_q[x]$.

2. Let $f \in \mathbf{Q}[x]$ be a polynomial of degree $n > 2$. Let K be the splitting field of f, and suppose that $\mathrm{Gal}(f) \simeq S_n$. Denote by $\alpha \in K$ a root of f.

(a) Show that f is irreducible over $\mathbf{Q}$.

(b) Show that the only field automorphism of $\mathbf{Q}(\alpha)$ is the identity.