DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
TUESDAY-AUGUST 28, 2001

Do five of the seven problems. All problems carry equal weight. Passing level: 60% with at least two problems substantially correct.

1. a) State the Bromwich-Mellin inversion formula for the Laplace transform, $\mathcal{L}$.

   b) Use the formula and contour integration techniques to evaluate the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{e^{-a\sqrt{s}}\} \qquad (a > 0)$$

2. Determine the eigenvalues λ and the eigenfunctions φ for the Sturm-Liouville problem:

$$-\frac{d^2\varphi}{dx^2} = \lambda\varphi \qquad (0 < x < 1)$$

$$\varphi'(0) = 0, \quad \varphi'(1) - h\varphi(1) = 0 \qquad (h > 0)$$

3. Display the Green's function for the BVP:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{1}{x^2}\right)y = f(x) \qquad (a < x < b)$$

$$y(a) = 0 = y(b)$$

with $0 < a < b < +\infty$.

4. Find the potential $u = u(r, \theta)$ in the sector:

$$\begin{cases} \Delta u = 0 & \text{in} \quad 0 < r < a,\ 0 < \theta < \alpha \\ u = 0 \ \text{ on} & \theta = 0, \alpha \quad (0 < r < a) \\ u = f(\theta) \ \text{ on} & r = a \quad (0 < \theta < \alpha) \end{cases}$$

5. A thin circular ring of radius a and thermal conductivity K is initially raised to a nonuniform temperature, and then it is left to cool. The governing equation is

$$\frac{\partial u}{\partial t} = \frac{K}{a^2} \frac{\partial^2 u}{\partial \theta^2} - H(u - U)$$

and the initial condition is $u(\theta, 0) = u_0(\theta)$.

a) Explain the physical meaning of the constants K, H and U.

b) Solve this IBVP by the Fourier series method.

6. At time $t = 0$, a string with ends fastened at $x = 0$ and $x = 1$ is plucked at the point $x = p$ $(0 < p < 1)$, and then released without initial velocity. The initial displacement $u(x, 0)$ is linear on $0 < x < p$ and $p < x < 1$, and $u(p, 0) = h$. Find the vibrations of the displacement $u(x, t)$ for $t > 0$ in a series form.

7. Consider a arbitrary function $f(x),\ \ 0 < x < 2\pi$, belonging to $L^2(0, 2\pi)$.

   a) Describe the concept of convergence of the Fourier series for f in the L^2 sense.

   b) Under what conditions on f does the Fourier series converge pointwise and uniformly for $0 \leq x \leq 2\pi$?

   c) Describe "Gibbs' Phenomenon," giving an example of a function f that would produce it, and explain its consequences for convergence in (a) and (b).