Do 7 of the following 9 problems. Indicate clearly which problems should be graded.

Passing Standard: For Master’s level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Part I  Linear Algebra

1. Let \( A \) be an \( n \times n \) matrix (over some field of scalars), and let \( c(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \) be its characteristic polynomial.
   (a) Prove that \( A \) is nonsingular if and only if \( c_0 \neq 0 \).
   (b) The Cayley–Hamilton Theorem says \( c(A) \) is equal to the 0 matrix. If \( A \) is nonsingular, use this to express \( A^{-1} \) as a polynomial in \( A \).

2. Determine whether or not the matrix \( A \) is diagonalizable over \( \mathbb{C} \):

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
3 & 0 & 1 \\
1 & -1 & 2
\end{bmatrix}
\]

[Hint: 2 and \(-1\) occur as eigenvalues.]

3. Let \( V \) be a finite-dimensional inner product space over \( \mathbb{R} \), with inner product \( \langle u, v \rangle \).
   (a) Prove that any set \( S \) of nonzero, pairwise orthogonal vectors is linearly independent.
   (b) If \( T : V \rightarrow V \) is a linear operator, satisfying \( \langle T(u), v \rangle = \langle u, T(v) \rangle \) for all \( u, v \), prove that the eigenvectors associated with distinct eigenvalues \( \lambda \) and \( \mu \) are orthogonal.

4. Let \( f : V \rightarrow F \) be a linear functional on a vector space \( V \) over a field \( F \), and let \( N \) be the nullspace (or kernel) of \( f \). Assume \( f \neq 0 \) so that \( N \neq V \). Fix a vector \( v_0 \in V - N \).
   (a) Given any \( v \in V \), show there exist \( c \in F \) and \( n \in N \) such that \( v = cv_0 + n \).
   (b) Show that the pair \((c, n)\) in (a) such that \( v = cv_0 + n \) is unique.
Part II  Advanced Calculus

1. Let \( f : (a, b) \to \mathbb{R} \) be uniformly continuous, with \((a, b)\) a finite open interval in \(\mathbb{R}\).
   
   (a) Prove that \( f \) is bounded.
   
   (b) Must this be true if \((a, b)\) is replaced by \(\mathbb{R}\)?
   
   (c) Must \( f \) be bounded if \( f \) is only assumed to be continuous on \((a, b)\)?

2. Let \( f : [0, 1] \to \mathbb{R} \) be monotone increasing: \( c < d \) implies \( f(c) \leq f(d) \).
   Use the definition of the Riemann integral (comparing upper and lower sums relative to a partition of \([0, 1]\)) to prove that \( \int_0^1 f(x) \, dx \) exists.

3. Compute \( I = \iiint_R \sqrt{x^2 + y^2} \, dV \), where \( R \) is the region bounded by the planes \( z = 0 \), \( z = 6 \), and the cylinder \( x^2 + y^2 = 4 \).

4. Let \( F \) be the vector field on \( \mathbb{R}^3 \) given by
   \[
   F(x, y, z) = x^2 y \vec{i} + z \vec{j} + xyz \vec{k} = (x^2 y, z, xyz).
   \]
   Is there a vector field \( G \) on \( \mathbb{R}^3 \) such that \( F = \nabla \times G = \text{curl } G \)? Discuss why or why not.

5. Find real numbers \( a, b \) for which
   \[
   \int_0^\pi \left[ \sin x - (ax^2 + bx) \right]^2 \, dx
   \]
   is minimized; thus \( ax^2 + bx \) is a least squares approximation of \( \sin x \) on \([0, \pi]\). You may differentiate under the integral sign and use the facts that \( \int_0^\pi x^2 \sin x \, dx = \pi^2 - 4 \) and \( \int_0^\pi x \sin x \, dx = \pi \).