Department of Mathematics and Statistics
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Basic Exam: Linear Algebra/Advanced Calculus

August 27, 2001

Do 7 of the following 9 problems. Indicate clearly which problems should be graded.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Part I Linear Algebra

1. Let A be an $n \times n$ matrix (over some field of scalars), and let $c(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be its characteristic polynomial.

   (a) Prove that A is nonsingular if and only if $c_0 \neq 0$.

   (b) The Cayley–Hamilton Theorem says $c(A)$ is equal to the 0 matrix. If A is nonsingular, use this to express A^{-1} as a polynomial in A.

2. Determine whether or not the matrix A is diagonalizable over $\mathbb{C}$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

   [Hint: 2 and -1 occur as eigenvalues.]

3. Let V be a finite-dimensional inner product space over $\mathbb{R}$, with inner product $\langle u, v \rangle$.

   (a) Prove that any set S of nonzero, pairwise orthogonal vectors is linearly independent.

   (b) If $T : V \to V$ is a linear operator, satisfying $\langle T(u), v \rangle = \langle u, T(v) \rangle$ for all u, v, prove that the eigenvectors associated with *distinct* eigenvalues λ and μ are orthogonal.

4. Let $f : V \to F$ be a linear functional on a vector space V over a field F, and let N be the nullspace (or kernel) of f. Assume $f \neq 0$ so that $N \neq V$. Fix a vector $v_0 \in V - N$.

   (a) Given any $v \in V$, show there exist $c \in F$ and $n \in N$ such that $v = cv_0 + n$.

   (b) Show that the pair (c, n) in (a) such that $v = cv_0 + n$ is unique.

Part II Advanced Calculus

1. Let $f : (a, b) \to \mathbb{R}$ be uniformly continuous, with (a, b) a finite open interval in $\mathbb{R}$.

   (a) Prove that f is bounded.

   (b) Must this be true if (a, b) is replaced by $\mathbb{R}$?

   (c) Must f be bounded if f is only assumed to be continuous on (a, b)?

2. Let $f : [0, 1] \to \mathbb{R}$ be monotone increasing: $c < d$ implies $f(c) \leq f(d)$. Use the definition of the Riemann integral (comparing upper and lower sums relative to a partition of $[0, 1]$) to prove that $\int_0^1 f(x)\, dx$ exists.

3. Compute $I = \iiint_R \sqrt{x^2 + y^2}\, dV$, where R is the region bounded by the planes $z = 0$, $z = 6$, and the cylinder $x^2 + y^2 = 4$.

4. Let F be the vector field on $\mathbb{R}^3$ given by

$$F(x, y, z) = x^2 y\vec{i} + z\vec{j} + xyz\vec{k} = (x^2 y, z, xyz).$$

   Is there a vector field G on $\mathbb{R}^3$ such that $F = \nabla \times G =$ curl G? Discuss why or why not.

5. Find real numbers a, b for which

$$\int_0^\pi [\sin x - (ax^2 + bx)]^2 dx$$

   is minimized; thus $ax^2 + bx$ is a least squares approximation of $\sin x$ on $[0, \pi]$. You may differentiate under the integral sign and use the facts that $\int_0^\pi x^2 \sin x\, dx = \pi^2 - 4$ and $\int_0^\pi x \sin x\, dx = \pi$.