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ADVANCED EXAM - DIFFERENTIAL EQUATIONS
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Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1. Consider the Cauchy problem for the forced wave equation in one dimension:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t),$$

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

   (a) Use Duhamel's principle to derive a formula for $u(x,t)$.

   (b) Find the solution u when the forcing is a harmonic point source with frequency ω moving with velocity v,

$$f(x,t) = \delta(x - v\,t)\ \sin \omega t, \quad \text{with} \quad 0 < v < c.$$

   Exhibit the solution only for $x > 0$, $\frac{x}{c} < t < \frac{x}{v}$.

   (c) Show that this demonstrates the Doppler effect: the harmonic oscillation at x over the time interval in which the source approaches x has frequency

$$\widetilde{\omega} = \frac{\omega}{1 - v/c} > \omega.$$

2. (a) Show that the Lorentz equations

$$\begin{aligned} \dot{x} &= -\sigma\,(x - y) \\ \dot{y} &= r\,x - y - x\,z \\ \dot{z} &= -b\,z + x\,y, \end{aligned}$$

   (where σ, r and b are positive constants), has a hyperbolic fixed point of saddle type at the origin $(0,0,0)$, provided $r > 1$.

(b) Characterize precisely the tangent space to the stable manifold at the origin, for $r > 1$.

3. Suppose $u \in H^1(\Omega)$ is a weak solution to

$$\Delta u = 0 \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a regular open subset of the plane R^2. Show that for any compact $K \subset \Omega$, there is a constant C (independent of u) such that

$$\int_K |\nabla u|^2 \, dx\, dy \leq C \int_\Omega u^2 \, dx\, dy.$$

4. Suppose that $f(x)$ is a smooth vector field on R^n, and that $f(0) = 0$. Let λ_i be the eigenvalues of $Df(0)$, and assume that $Re\, \lambda_i < 0$ for $1 \leq i \leq n$. Find $\delta > 0$ and $\alpha > 0$ so that every solution $x(t)$ of

$$\frac{dx}{dt} = f(x)$$

with $|x(0)| < \delta$ satisfies

$$|x(t)| < \delta\, e^{-\alpha t}$$

for $t \geq 0$. [Give details - don't just cite a theorem.]

5. Consider the system

$$\begin{aligned} x' &= -\lambda(x^2+y^2)\, x + \omega(x^2+y^2)\, y \\ y' &= -\omega(x^2+y^2)\, x - \lambda(x^2+y^2)\, y \end{aligned}$$

where $\lambda(r)$ and $\omega(r)$ are given smooth functions of $r \geq 0$.

(a) Determine whether the rest point $(x, y) = (0, 0)$ is stable or unstable in terms of $\lambda(0)$, whenever $\lambda(0) \neq 0$. What is the qualitative behavior when $\lambda(0) < 0$ and $\omega(0) > 0$?

(b) Suppose now that

$$\lambda(r) = r\,(1-r)\,(2-r) \quad \text{and} \quad \omega(r) = (\frac{1}{2} - r)\,(\frac{3}{2} - r).$$

Show that every ω-limit set of a point $(\bar{x}, \bar{y})$ is either the rest point $(0, 0)$ or a periodic orbit lying on one of the two circles

$$x^2 + y^2 = 1 \quad \text{or} \quad x^2 + y^2 = 2.$$

[Hint: Consider a suitable Liapunov function.]

6. Suppose $u \in R^3$ is a classical solution of the linear elasticity equations,

$$m\,u_{tt} - \mu\,\Delta u - (\lambda + \mu)\,\text{grad}(\text{div}u) = 0,$$

with initial data having compact support. Show that the energy

$$E(t) = \int [m\,u_t^2 + \mu\,(\text{grad}u)^2 + (\lambda + \mu)\,(\text{div}u)^2]\,dx$$

is conserved, and use this to show that solutions are unique.

7. Suppose that $u(x, t)$ solves the heat equation in the parabolic cylinder U_T. Show that

$$v(x, t) = u_t^2 + |\nabla u|^2$$

is a subsolution, *i.e.* satisfies

$$v_t - \Delta v \leq 0$$

in U_T.