Do five of the following problems. All problems carry equal weight.

Passing level:
Masters: 60% with at least two substantially correct.
Ph.D: 75% with at least three substantially correct.

1. Derive a numerical differentiation scheme of the following form:

\[ f''(t) \approx A f(t + 2h) + B f(t + h) + C f(t) \]

Also derive a formula for the error in making this approximation.

2. For solving \( y' = f(x, y) \), consider the numerical method

\[ y_{n+1} = y_n + \frac{h}{2} (y'_n + y'_{n+1}) + \frac{h^2}{12} (y''_n - y''_{n+1}), \]

where \( n = 0, 1, \ldots \), and \( h = x_{n+1} - x_n \) (the step size).

(a). Show that this method is at least fourth order accurate.

(b). For the equation \( y' = \lambda y, \ y(0) = \epsilon \neq 0 \), show that the method will not blow up if \( \lambda \) is negative and real as \( n \rightarrow \infty \).

3. Consider the numerical integration rule

\[ \int_{-1}^{1} f(x) \, dx = w_1 f(-x_1) + w_0 f(0) + w_1 f(x_1). \]

(a). Write down and solve the conditions for \( x_1, w_0 \) and \( w_1 \).

(b). Use this rule to derive an approximation for the general integral \( \int_a^b f(x) \, dx \).
4(a). Show that if \( \|A\| < 1 \) in some induced matrix norm, then the iteration

\[
v^{k+1} = A v^k + b
\]

converges where \( A \) is any \( n \) by \( n \) matrix; \( b \) and \( v^k \) are \( n \)-vectors.

(b). Show that if \( A \) is a strictly diagonally dominant matrix, then Jacobi iteration for the linear system

\[
A x = b
\]

converges.

5. Define an iteration formula by

\[
x_{n+1} = z_{n+1} - \frac{f(z_{n+1})}{f'(x_n)}, \quad z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

for finding the simple root \( \alpha \) of \( f(x) \). Show that the order of convergence is at least 2. To make your final conclusion you may quote a well-known theorem.

6. Let \( p_n(x) \) be the interpolation polynomial of degree \( \leq n \) interpolating \( f(x) = e^x \) at the points \( x_i = \frac{i}{n} \), \( i = 0, 1, 2, \ldots, n \).

(a). Derive a good upper bound for

\[
\max_{0 \leq x \leq 1} |e^x - p_n(x)|
\]

using the hint below, and determine the smallest \( n \) guaranteeing an error less than \( 10^{-2} \) on \([0, 1]\).

[Hint: First show that for any integer \( i \) with \( 0 \leq i \leq n \), one has

\[
\max_{0 \leq x \leq 1} \left| \frac{1}{n} - \frac{i}{n} \right| \left( \frac{x}{n} - \frac{n-i}{n} \right) \leq \frac{1}{4} \]

(b). Solve the analogous problem for the \( n \)th-degree Taylor polynomial for \( f(x) \) and compare the result with the one in (a).
7. We wish to interpolate the function \( f(x) \) at the points

\[
x_j = j \frac{2\pi}{2n+1}, \quad j = 0, \ldots, 2n
\]

for some positive integer \( n \) by a polynomial of the form

\[
\sum_{k=-n}^{n} c_k e^{ikx}
\]

i.e., we want

\[
\sum_{k=-n}^{n} c_k e^{ikx_j} = f(x_j), \quad j = 0, 1, \ldots, 2n.
\] (1)

Use the relation

\[
\sum_{j=0}^{2n} e^{ikx_j} = \begin{cases} 
2n + 1 & \text{if } k \text{ is an integer multiple of } 2n+1 \\
0 & \text{if } k \text{ is any other integer}
\end{cases}
\]

for \( k = 0, 1, \ldots, 2n \) to derive expressions for \( c_k \). Hint: Multiply equation (1) by \( e^{-i\ell x_j} \) and sum appropriately where \( \ell \) is an integer which satisfies \( -n \leq \ell \leq n \). Here \( i = \sqrt{-1} \).