DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM – NUMERICS
AUGUST 29, 2000

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correct.
Ph.D: 75% with at least three substantially correct.

1. Derive a numerical differentiation scheme of the following form:

$$f''(t) \approx Af(t+2h) + Bf(t+h) + Cf(t)$$

Also derive a formula for the error in making this approximation.

2. For solving $y' = f(x, y)$, consider the numerical method

$$y_{n+1} = y_n + \frac{h}{2}(y'_n + y'_{n+1}) + \frac{h^2}{12}(y''_n - y''_{n+1}),$$

where $n = 0, 1, ...,$ and $h = x_{n+1} - x_n$(the step size).

(a). Show that this method is at least fourth order accurate.

(b). For the equation $y' = \lambda y$, $y(0) = \epsilon \neq 0$, show that the method will not blow up if λ is negative and real as $n \to \infty$.

3. Consider the numerical integration rule

$$\int_{-1}^{1} f(x)\, dx = w_1\, f(-x_1) + w_0\, f(0) + w_1\, f(x_1).$$

(a). Write down and solve the conditions for x_1, w_0 and w_1.

(b). Use this rule to derive an approximation for the general integral $\int_a^b f(x)\, dx$.

4(a). Show that if $\|A\| < 1$ in some induced matrix norm, then the iteration

$$v^{k+1} = A\, v^k + b$$

converges where A is any n by n matrix; b and v^k are n-vectors.

(b). Show that if A is a strictly diagonally dominant matrix, then Jacobi iteration for the linear system

$$A\, x = b$$

converges.

5. Define an iteration formula by

$$x_{n+1} = z_{n+1} - \frac{f(z_{n+1})}{f'(x_n)}, \quad z_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

for finding the simple root α of $f(x)$. Show that the order of convergence is at least 2. To make your final conclusion you may quote a well-known theorem.

6. Let $p_n(x)$ be the interpolation polynomial of degree $\leq$ n interpolating $f(x) = e^x$ at the points $x_i = \frac{i}{n}$, $i = 0, 1, 2, \ldots, n$.

(a). Derive a good upper bound for

$$\max_{0 \leq x \leq 1} |e^x - p_n(x)|$$

using the hint below, and determine the smallest n guaranteeing an error less than 10^{-2} on $[0, 1]$. [Hint: First show that for any integer i with $0 \leq i \leq n$, one has

$$\max_{0 \leq x \leq 1} |(x - \frac{i}{n})(x - \frac{n-i}{n})| \leq \frac{1}{4}.]$$

(b). Solve the analogous problem for the nth-degree Taylor polynomial for $f(x)$ and compare the result with the one in (a).

7. We wish to interpolate the function $f(x)$ at the points

$$x_j = j\frac{2\pi}{2n+1}, \;\; j = 0, ..., 2n$$

for some positive integer n by a polynomial of the form

$$\sum_{k=-n}^{n} c_k e^{ikx}$$

i.e., we want

$$\sum_{k=-n}^{n} c_k e^{ikx_j} = f(x_j), \quad j = 0, 1, ..., 2n. \tag{1}$$

Use the relation

$$\sum_{j=0}^{2n} e^{ikx_j} = \begin{cases} 2n+1 & \text{if } k \text{ is an integer multiple of 2n+1} \\ 0 & \text{if } k \text{ is any other integer} \end{cases}$$

for $k = 0, 1, ..., 2n$ to derive expressions for c_k. Hint: Multiply equation (1) by $e^{-i\ell x_j}$ and sum appropriately where ℓ is an integer which satisfies $-n \leq \ell \leq n$. Here $i = \sqrt{-1}$.